UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 28 May, 2010 9:00 am to 11:00 am

PAPER 25

TOPICS IN ANALYTIC NUMBER THEORY

*Attempt no more than **TWO** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

This question is about the averages of various arithmetic functions. The divisor function τ is defined by letting $\tau(n)$ be the number of natural numbers dividing n; the Möbius function μ is defined by

$$\mu(n) \;=\; \begin{cases} (-1)^k & \text{if } \; n = p_1 \,\ldots\, p_k \;\text{ with }\; p_1, \,\ldots\,, p_k \;\text{ distinct primes}\,, \\ 0 & \text{otherwise}\,; \end{cases}$$

and the von Mangoldt function Λ is defined by

$$\Lambda(n) \;=\; \begin{cases} \log p & \text{if } \; n = p^k \;\text{ with }\; p \text{ prime}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Write

$$\Delta(n) \;:=\; \sum_{x \leqslant n} \left(\tau(x) - \log x - 2\gamma\right),$$

where

$$\gamma \;:=\; \int_1^\infty \frac{\{x\}}{x\lfloor x \rfloor}\, dx\,.$$

Prove that $\Delta(n) = O(\sqrt{n})$. Hence or otherwise prove that if

$$\frac{1}{n}\sum_{x \leqslant n} \mu(x) \;=\; o(1)$$

then

$$\frac{1}{n}\sum_{x \leqslant n} \Lambda(x) \;=\; 1 + o(1)\,.$$

Take care to prove any identities between arithmetic functions that you use.

2

This question concerns a proof of the prime number theorem. Suppose that $f \in \ell^\infty(\mathbb{N})$ and $\sigma > 1$ is a damping constant. We write $m_{f,\sigma}$ for the (complex) measure on $\mathbb{R}$ induced by the map

$$g \mapsto \sum_{n \in \mathbb{N}} f(n)\, n^{-\sigma} g(\log n)\,.$$

Writing 1 for the constant function taking the value 1 and μ for the Möbius function we have

$$\widehat{m_{1,\sigma}}(t) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+it}} \quad \text{and} \quad \widehat{m_{\mu,\sigma}}(t) \;=\; \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\sigma+it}}\,.$$

Prove that

$$|\widehat{m_{\mu,\sigma}}(t)| \;=\; O((\sigma-1)^{-3/4} \log^{1/2}(2+|t|))\,.$$

Give a short description of how this may be used to prove

$$\frac{d^k\, \widehat{m_{\mu,\sigma}}}{dt^k}(t) \;=\; O_k((\sigma-1)^{-3(k+1)/4} \log^{O(k)}(2+|t|))$$

for any natural number $k \in \mathbb{N}$. Using this estimate, or otherwise, prove that there is some absolute $c > 0$ such that

$$\sum_{x \leqslant n} \mu(x) \log^k x \;=\; O_k(n \log^{(1-c)\,k+O(1)} n)\,.$$

You may assume that for any $x \in \mathbb{R}$ one has

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp(x(\sigma+it))}{(\sigma+it)^2}\, dt \;=\; \begin{cases} x & \text{if } x \geqslant 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

3

This question is about Dirichlet's theorem on primes in arithmetic progressions. For $\Re s > 1$ (here $\Re s$ denotes the real part of s) the Riemann ζ-function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}\,.$$

Similarly, given a Dirichlet character χ the corresponding Dirichlet L-function is defined by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

in the same region. Write down the Euler product formula for $\zeta(s)$ and $L(s,\chi)$ valid in this region. Explain why $L(s,\chi) \neq 0$ in this region.

Prove Dirichlet's theorem that given coprime naturals a and q there are infinitely many primes p with $p \equiv a \pmod{q}$. You may assume the inversion formula:

$$\frac{1}{\phi(q)} \sum_{\chi} \overline{\chi(a)}\,\chi(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod{q}\,, \\ 0 & \text{otherwise}\,; \end{cases}$$

the orthogonality relation

$$\sum_{1 \leqslant n \leqslant q} \chi(n) = \begin{cases} \phi(q) & \text{if } \chi = \chi_0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where χ_0 is the principal character; and that ζ may be meromorphically continued to the range $\Re s > 0$ such that

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \{x\}\, x^{-s-1}\, dx\,.$$

4

This question concerns estimating the fractional part of αn^2. State Dirichlet's pigeon-hole principle which you may assume. Prove that if $\theta \in \mathbb{R}$ and $s, t \in \mathbb{Z}$ then we have the estimate

$$\left|\sum_{n=s}^{t} \exp(2\pi i\,\theta n)\right| \leqslant \min\left\{t-s+1\,,\ \frac{1}{2\|\theta\|}\right\}.$$

Prove that if $\theta_1, \ldots, \theta_k$ are δ-separated, i.e. $\|\theta_i - \theta_j\| \geqslant \delta$ for all $i \neq j$, then

$$\sum_{i=1}^{k} \min\left\{Q,\ \frac{1}{2\|\theta_i\|}\right\} = O((Q+\delta^{-1})\log Q).$$

State and prove Weyl's inequality giving an upper estimate for the sum

$$\left|\sum_{n=0}^{N} \exp(2\pi i\,\alpha n^2)\right|$$

when $|\alpha - a/q| \leqslant 1/qQ$. Assuming, if you wish, that for any prime P and set $A \subset G := \mathbb{Z}/P\mathbb{Z}$ of density α with $A \cap [-L, L] = \emptyset$, we have

$$\sup_{0 \neq |r| \leqslant (P/L)^2} \left|\sum_{x \in A} \exp(2\pi i\, rx/p)\right| \geqslant \alpha L/2P.$$

Prove that there is an absolute constant $c > 0$ such that

$$\min\{\|\alpha n^2\| : 1 \leqslant n \leqslant N\} = O(N^{-c})$$

where $\|\theta\| := \min\{|\theta - z| : z \in \mathbb{Z}\}$.

5

This question concerns the proof of Vinogradov's three primes theorem via Vaughn's identity. This identity is the decomposition

$$\widehat{\Lambda_N}(\theta) \;=\; S_1 + S_4 + S_5 + O(X)$$

where

$$\widehat{\Lambda_N}(\theta) \;=\; \sum_{n \leqslant N} \Lambda(n)\, \exp(2\,\pi i\,\theta n)\,,$$

$$S_1 \;:=\; \sum_{a \leqslant X} \mu(a) \sum_{b \leqslant N/a} \log b\, \exp(2\,\pi i\, ab\,\theta)\,,$$

$$S_4 \;=\; -\sum_{X < u \leqslant N}\; \sum_{a \,|\, u,\, a \leqslant X} \mu(a) \sum_{X < d \leqslant N/u} \Lambda(d)\, \exp(2\,\pi i\, ud\theta)\,,$$

and

$$S_5 \;:=\; -\sum_{a \leqslant X} \mu(a) \sum_{v \leqslant N/a}\; \sum_{d \leqslant \min\{X, N/av\}} \Lambda(d)\, \exp(2\,\pi i\, avd\theta)\,.$$

Prove that if $\theta \in \mathbb{R}$ has $|\theta - a/q| \leqslant 1/qQ$ for some naturals a and q with $i\,1 \leqslant q \leqslant Q$ and $X < \sqrt{N}$ then

$$S_4 \;=\; O\Big(\Big(N/\sqrt{q} \,+\, N/\sqrt{X} \,+\, \sqrt{Nq}\Big) \log^{O(1)} N\Big)\,.$$

You may assume, if you wish, that

$$\sum_{x \leqslant n} \tau(x)^2 \;=\; O(n\; log^3 n)\,;$$

that if $i\,\alpha \in \mathbb{R}$ and $t \geqslant s$ are integers then we have the estimate

$$\left|\sum_{n=s}^{t} \exp(2\,\pi i\,\alpha n)\right| \;\leqslant\; \min\left\{t - s + 1,\; \frac{1}{2\|\alpha\|}\right\}$$

where $\|\alpha\| := \min\{|\alpha - z| : z \in \mathbb{Z}\}$; and that

$$\sum_{r \leqslant M} \min\left\{R,\, \frac{1}{\|\theta r\|}\right\} \;=\; O\Big(\big(MR/q \,+\, R \,+\, q \,+\, M\big) \log R\Big)\,.$$

State the Siegel-Walfisz theorem. Show that for $i\,q \leqslant \log^A N$ and any $B > 0$ we have

$$\widehat{\Lambda_N}(a/q) \;=\; \frac{\mu(q)}{\phi(q)}\, N \,+\, O_{A,B}\Big(N\; log^{-B} N\Big)\,.$$

You may assume the Möbius inversion formula.

END OF PAPER