UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 27 May, 2010 9:00 am to 11:00 am

PAPER 24

LOCAL FIELDS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) What does it mean for absolute values on a field K to be equivalent? Let L/K be an extension of fields and $|\cdot|$ an absolute value on L. Show that $|\cdot|$ is non-archimedean if and only if its restriction to K is non-archimedean.

(b) Classify the non-archimedean absolute values on $\mathbb{Q}$ up to equivalence. Explain with proof how this result generalises to number fields. [*Standard facts about Dedekind domains and DVR's may be assumed.*]

2

(a) Let $(K, |\cdot|)$ be a complete non–archimedean valued field. Let L/K be a finite extension. Show that there is at most one absolute value on L extending $|\cdot|$ on K.

(b) Let K be the splitting field of a polynomial in $\mathbb{Q}_p[X]$, say with roots $\alpha_1, \ldots, \alpha_n$. Let $\mathcal{O}_K$ be the valuation ring of K. Which of the following statements are true and which are false? Justify your answers.

(i) If $x \in K$ with $\mathrm{Tr}_{K/\mathbb{Q}_p}(x) \in \mathbb{Z}_p$ then $x \in \mathcal{O}_K$.

(ii) If $x \in K$ with $N_{K/\mathbb{Q}_p}(x) \in \mathbb{Z}_p^*$ then $x \in \mathcal{O}_K^*$.

(iii) If $\beta \in K$ with $|\beta - \alpha_1| < |\beta - \alpha_i|$ for all $i = 2, \ldots, n$ then $\alpha_1 \in \mathbb{Q}_p(\beta)$.

(iv) If $\beta \in K$ with $|\beta - \alpha_1| < |\beta - \alpha_i|$ for all $i = 2, \ldots, n$ then $\beta \in \mathbb{Q}_p(\alpha_1)$.

3

(a) State a version of Hensel's lemma, and use it to show that there is a unique prime p for which $Q(x, y, z) = 5x^2 + 7y^2 + 13z^2 = 0$ has no non-trivial solutions over $\mathbb{Q}_p$. [*It may help to note that* $Q(2, 1, 1) = 2^3\, 5$.]

(b) Let $(K, |\cdot|)$ be a complete non-archimedean valued field with residue field k. Show that if K is locally compact then $|\cdot|$ is discrete and k is finite.

(c) Classify the non-archimedean local fields of characteristic $p > 0$.

4

(a) Let L/K be finite extensions of $\mathbb{Q}_p$ with residue fields k_L and k. Let $\alpha \in \mathcal{O}_L$ with $k_L = k(\overline{\alpha})$. Show that if L/K is unramified then $\mathcal{O}_L = \mathcal{O}_K[\alpha]$.

(b) Define the different $\mathcal{D}_{L/K}$. Show that if $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ then $\mathcal{D}_{L/K} = (g'(\alpha))$ where g is the minimal polynomial of α over K.

(c) Compute $\delta(L/K) = v_L(\mathcal{D}_{L/K})$ for L/K the field extensions $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$ and $\mathbb{Q}_p(\zeta_{p^2-1})/\mathbb{Q}_p$. [*Here ζ_m is a primitive mth root of unity, and v_L is the normalised discrete valuation on L.*]

5

(a) Write an essay on higher ramification groups.

(b) Let L be the splitting field of $f(X) = X^3 + pX + p$ over $\mathbb{Q}_p$. Determine $G = \mathrm{Gal}(L/\mathbb{Q}_p)$ and the ramification groups $(G_n)_{n \geqslant 0}$ for every odd prime p. [*The discriminant of $X^3 + rX + s$ is $-4r^3 - 27s^2$.*]

END OF PAPER