UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2010 9:00 am to 12:00 pm

PAPER 23

IWASAWA THEORY OF ELLIPTIC CURVES WITH COMPLEX MULTIPLICATION

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let k be a field, and E an elliptic curve defined over k. Let l be a prime number distinct from the characteristic of k. Define the l-adic spaces $T_l(E)$ and $V_l(E)$. Suppose we are given an injection

$$\theta : K \rightarrow End_k(E) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} ,$$

where K is an imaginary quadratic field. We identify K with $\theta(K)$, and define

$$K_l = K \underset{\mathbb{Q}}{\otimes} \mathbb{Q}_l \ , \ R = K \cap End_k(E) \ , \ R_l = R \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_l \ .$$

Prove that (i) $V_l(E)$ is a free K_l-module of rank 1, (ii) an element ϕ of K_l maps $T_l(E)$ into itself if and only if ϕ belongs to R_l, and (iii) the image of the absolute Galois graph of k in the group of $\mathbb{Z}_l$ – automorphisms of $T_l(E)$ is abelian.

2

Let p be any prime number, and let I be the ring of integers of the completion of the maximal unramified extension of $\mathbb{Q}_p$. Let G be a profinite abelian group. Define what is meant by an I-valued measure on G. Define the Iwasawa algebra of G with coefficients in I, and show that it can be identified with the ring of I-valued measures on G.

3

Let E be the elliptic curve over $\mathbb{Q}$ defined by

$$E \ : \ y^2 \ = \ x^3 \ - \ x \ .$$

Prove that $End_{\mathbb{Q}}(E) = \mathbb{Z}$. Let $K = \mathbb{Q}(i)$, where $i^2 = -1$. Prove that $End_K(E) = \mathbb{Z}[i]$. Let $\gamma_2 = (1+i)\mathbb{Z}[i]$. Show that no fourth root of unity, distinct from 1, is congruent to $1 \, mod \, \gamma_2^3$. Write down explicity the Grossencharacter of E over K. If γ is any prime ideal of K distinct from γ_2, and $\pi = \psi_E(\gamma)$, where ψ_E is the Grossencharacter of E over K, let $[\pi](t)$, where $t = -x/y$, be the endomorphism of the formal group of E at γ defined by π. Prove that

$$[\pi](t) \ \equiv \ t^{N\gamma} \, mod \, \gamma \ ,$$

where $N\gamma$ is the cardinality of $\mathbb{Z}[i]/\gamma$.

4

Write an essay on the proof of the existence of a Grossencharacter for any elliptic curve with complex multiplication, defined over a number field.

5

Write an essay on the one variable main conjecture for elliptic curves with complex multiplication, discussing, in particular, the construction of the relevant p-adic L-function.

END OF PAPER