UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 27 May, 2010 9:00 am to 12:00 pm

PAPER 21

CATEGORY THEORY

*Attempt **ONE** question from Section A and **TWO** questions from Section B.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

State and prove *either* the General *or* the Special Adjoint Functor Theorem. Use it to show that the inclusion functor from the category of compact Hausdorff spaces to that of all topological spcaes has a left adjoint.

[Standard results from general topology may be assumed. You may also assume the result that if a Hausdorff space X has a dense subspace of cardinality K, then the cardinality of X is at most 2^{2^K}.]

2

"The Yoneda Lemma tells us that every locally small category is equivalent to a full subcategory of a functor category $[C, Set]$. Thus category theory is reduced to the study of such functor categories."

Write a short essay arguing the case *either* for *or* against this assertation.

SECTION B

3

Explain what is meant by the terms *balanced category*, *strong monomorphism* and *regular monomorphism.* If every monomorphism in C is strong, show that C is balanced. Conversely, if C is balanced and has pullbacks, show that every monomorphism in C is strong. [You may assume the result that any pullback of a monomorphism is monic.]

Now suppose that not every strong monomorphism in C is regular. Show that there is a faithful functor $C_0 \to C$, where C_0 is a certain finite category (to be determined) with four objects and six non-identity morphisms. Show also that C_0 contains a strong monomorphism which is not regular.

4

Let $G : D \to C$ be a functor having a left adjoint F. Explain how to define the *comparison functor* $K : D \to C^{\mathbb{T}}$, where $\mathbb{T}$ is the monad on C induced by $(F \dashv G)$, and its left adjoint L in the case when D has coequalizers. [Detailed proofs are not required.]

Now let n be a positive integer, and consider the category C_n whose objects are sets A equipped with n partial unary operations $(\alpha_1, \alpha_2, \dots, \alpha_n)$, such that $\alpha_1(x)$ is defined for all $x \in A$, and for each $i > 1$ $\alpha_i(x)$ is defined if and only if ($\alpha_{i-1}(x)$ is defined and) $\alpha_{i-1}(x) = x$. Morphisms of C_n are functions f satisfying $f(\alpha_i(x)) = \alpha_i(f(x))$ whenever $\alpha_i(x)$ is defined.

(i) Show that the forgetful functor $C_{n+1} \to C_n$ which 'forgets' the operation α_{n+1} has a left adjoint.

(ii) Assuming the result that the forgetful functor in (i) is monadic, show that the forgetful functor $C_n \to Set$ has monadic length n.

5

A functor $F : C \to D$ is called *final* if, for each object b of D, the arrow category $(b \downarrow F)$ is (nonempty and) connected. If F is final, show that for any $G : D \to E$ each cone under $G \circ F$ has a unique extension to a cone under G, and deduce that if E has colimits of shape C then it also has colimits of shape D.

A functor $G : D \to E$ is called a *discrete fibration* if, given $b \in ob\ D$ and $f : c \to Gb$ in E, there exists a unique $g : \bar{c} \to b$ in D with $Gg = f$. Show that an arbitrary functor $H : C \to E$ can be factored as

$$C \xrightarrow{F} D \xrightarrow{G} E$$

where F is final and G is a discrete fibration. [Hint: take the objects of D to be connected components of the categories $(c \downarrow H)$, $c \in obE$.]

6

Define the notions of *regular category*, and of *capital regular category*. Assuming the result that every small regular category admits an isomorphism-reflecting regular functor to a small capital regular category, prove that every small regular category admits an isomorphism-reflecting regular functor to a power of Set. State and prove an elementary condition which is necessary and sufficient for a regular category to admit an isomorphism-reflecting regular functor to Set.

END OF PAPER