UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 4 June, 2010 9:00 am to 12:00 pm

PAPER 20

SET THEORY AND LOGIC

*Attempt no more than **FOUR** questions.*

*There are **NINE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Write an essay on computable functions $\mathbb{N}^k \rightharpoonup \mathbb{N}$.

2

Prove the theorem of Gale and Stewart that every game of length ω over an arbitrary arena with open payoff set is determined. Indicate how your analysis cannot be extended to cover games with arbitrary payoff set unless AC fails.

3

Prove that every filter on a set can be extended to an ultrafilter. State and prove Łoś's theorem. Use it to prove that if every finite subset of a theory T in a countable first-order language has a model so does T itself.

4

(a) Prove the lemma of Craig that any theory with a semidecidable axiomatisation also has a decidable axiomatisation.

(b) Let A be a propositional formula, and 'p' a letter occurring in A. Show how to find A' and A'' neither containing occurrences of 'p' such that A is logically equivalent to $(A' \wedge p) \vee (A'' \wedge \neg p)$. Hence or otherwise prove that if A and B are two propositional formulæ with $A \vdash B$ then there is a formula C containing only those letters common to both A and B, such that $A \vdash C$ and $C \vdash B$.

5

Explain the device of Rieger-Bernays permutation models, and use it to prove the independence of the axiom of foundation from ZF. Extend your technique to prove the independence of the axiom of choice from ZF minus foundation.

6

Use Ramsey's theorem to prove (i) the Ehrenfeucht-Mostowski theorem, and (ii) the consistency of simple typed set theory with typical ambiguity and *urelemente*.

7

State and prove the Erdös-Rado theorem for exponent 2 on the existence of uncountable monochromatic sets. One consequence of this theorem is that a certain increasing function on ordinals is total. Give a condition (the *tree property*) for a supremum of iterates of this function to be a fixed point for it. Prove that any cardinal with the tree property must be strongly inaccessible.

8

Give a proof of the Ehrenfeucht-Mostowski theorem using ultraproducts not Ramsey's theorem.

9

Prove that elementarily equivalent structures have isomorphic ultralimits.

END OF PAPER