UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2010 9:00 am to 12:00 pm

PAPER 2

TOPICS IN REPRESENTATION THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let k be an algebraically closed field and let G be a finite group.

Prove that the number of isomorphism classes of simple kG-modules is equal to the number of conjugacy classes of elements of G of order not divisible by the characteristic of k.

Let H be as subgroup of G and let M be a kG-module. Define what is meant for M to be relatively H-projective. Show that if the index of H in G is invertible in k then M is relatively H-projective.

2

Let k be a field and let A be a finite dimensional k-algebra. Explain why A is a direct sum of indecomposable projective left A-modules, each generated by a primitive idempotent.

Define what is meant for A to be (a) Frobenius, (b) symmetric.

Let M be a finitely generated left A-module. Define $\mathrm{Soc}(M)$ and $\mathrm{Rad}(M)$.

Suppose A is symmetric. Let P be an indecomposable projective left A-module. Prove that $\mathrm{Soc}(P)$ is isomorphic to $P/\mathrm{Rad}(P)$.

Discuss the representation theory of kS_3 in the cases where k is algebraically closed of characteristic 2 or 3.

3

Let k be an algebraically closed field and let A be a finite dimensional k-algebra.

What does it mean for A to be (a) basic, (b) hereditary? Define the Ext quiver Q of A.

Suppose A is basic and hereditary. Prove that it is isomorphic to the path algebra kQ. Explain why the blocks of A correspond to the components of the underlying graph of Q.

4

Let k be an algebraically closed field and let Q be a quiver.

What does it mean for the path algebra kQ to be of finite representation type?

Give, with justification, examples of

(i) a quiver Q_1 with kQ_1 finitely generated as a k-algebra, but not finite dimensional as a k-vector space,

(ii) a quiver Q_2 with kQ_2 finite dimensional, but not of finite representation type.

Let Q_3 be a quiver whose underlying graph is the Dynkin diagram E_6. Sketch the proof that kQ_3 is of finite representation type.

5

Let G be a group and M be a left $\mathbb{Z}G$-module.

Define the cohomology groups $H^i(G, M)$ and write an essay describing the alternative descriptions of these groups in the cases where $i = 0,\ 1$ or 2. Your essay should include discussion of the example where G is a free abelian group of rank 2 and M is the trivial module $\mathbb{Z}$.

6

Let k be a field. Prove that a simple Artinian k-algebra A is isomorphic to a matrix algebra over a division algebra.

What does it mean for A to be a central simple k-algebra?

Define the Brauer group $B(k)$.

Show that (i) $B(\mathbb{Q})$ is an infinite group, and (ii) $B(\mathbb{R})$ is of order 2.

Explain briefly how $B(k)$ may be interpreted in terms of certain cohomology groups.

END OF PAPER