UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2010 1:30 pm to 4:30 pm

PAPER 19

SPECTRAL GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let p be a point on a Riemannian manifold M. State how the value at p of the Laplacian acting on a function on M may be calculated using suitable geodesics through p.

Let S^d be the unit sphere in $\mathbb{R}^{d+1}$ with the induced metric. State and prove the relation between the action of the Laplacian $\tilde{\Delta}$ on functions in $\mathbb{R}^{d+1}$ and that of the Laplacian Δ on the restrictions of those functions to S^d.

Derive the spectrum of the Laplacian acting on functions in S^d.

2

State and prove the transplation theorem for eigenfunctions of the Laplacian between manifolds constructed from copies of a Euclidean domain by identifying various pairs of boundary faces of those domains.

Construct, with proof, a pair of isospectral surfaces with boundary, one of which is simply-connected and orientable and the other of which has neither of these properties.

[*You may assume any version of the reflection principle that you require.*]

3

Define the heat operator L and heat kernel for a Riemannian manifold, M. State what this kernel, $g(p,q,t)$, is for the standard Euclidean space $\mathbb{R}^d$.

On a suitable neighbourhood, which you should specify, of the diagonal in $M \times M$ prove the existence of $\mathscr{C}^\infty$-functions w_k, $k = 0, 1, \ldots n, \ldots$, such that for

$$S_k(p,q,t) = g(p,q,t) \sum_{i=0}^{k} t^i w_i(p,q)\,,$$

$$LS_k = -g(p,q,t) t^k \Delta_q w_k\,,$$

where Δ_q is the Laplacian with respect to the q-coordinates on M.

Define a *parametrix* for the heat operator and give an explicit expression for one on M. Give an expression for the global heat kernel on M in terms of this parametrix.

4

Define a nowhere homogeneous or 'bumpy' metric on a differential manifold M^m and state Sunada's Lemma.

State the dimension of the k-jet space for maps from M^m to another differential manifold N^n, and identify its fibre as a bundle over $M \times N$.

Given a countable basis $\{U_i | i \in \mathbb{N}\}$ of open sets of M^m such that each $\overline{U}_i$ is diffeomorphic with a closed m-ball, for $i \neq j$ let $\mathscr{S}_{ij}$ be the set of metrics on M such that there is an isometry $\phi : \overline{U}_i \to \overline{U}_j$. Prove that the complement $\mathscr{C}\mathscr{S}_{ij}$ of $\mathscr{S}_{ij}$ is dense in the space of all metrics on M with the $\mathscr{C}^\infty$-topology.

5

State Sunada's Theorem for producing a pair of isospectral Riemannian manifolds with a common Riemannian covering. State a necessary condition for these manifolds not to be isometric.

A certain group T of order 96 has non-conjugate Gassmann-equivalent subgroups of index 12. T is generated by two elements s and d of order 3 with product of order 6, whose cube is central and does not lie in either subgroup. Use these data to construct an infinite sequence of pairs of Riemannian metrics (g_{1k}, g_{2k}), $k = 1, 2, \ldots$, on the topological surface of N of genus 2 such that:

1) no two distinct surfaces (N, g_{ik}), $(N, g_{j\ell})$ are isometric;

2) no two distinct surfaces are isospectral except that, for each k, (N, g_{1k}) and (N, g_{2k}) are isospectral;

3) each sequence of metrics g_{ik}, $i = 1, 2, 3 \ldots$, converges as $k \to \infty$ in the $\mathscr{C}^\infty$-topology to the hyperbolic metric g.

END OF PAPER