UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 1 June, 2010 1:30 pm to 4:30 pm

PAPER 17

ALGEBRAIC TOPOLOGY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

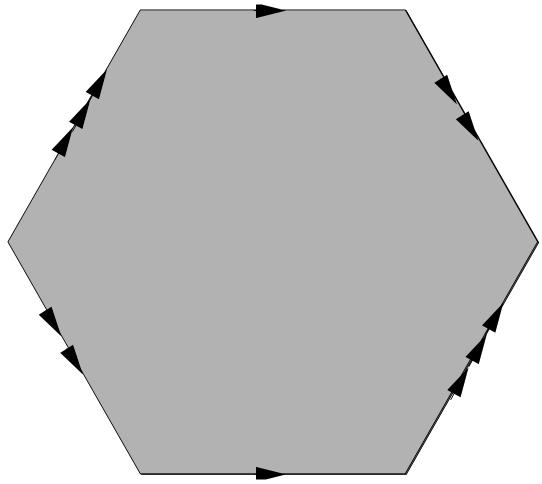
SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Compute $H_*(X)$ for the following spaces X:

(a) X is the space obtained by identifying opposite sides of a regular hexagon (the closure of the shaded region) as shown in the figure.



(b) X is the complement of S^m in S^n $(m < n)$, where

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$$

$$S^m = \{(x_1, \dots, x_{m+1}, 0, \dots, 0) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{m+1} x_i^2 = 1\}$$

(c) $X = T^2 \times [0,1]/\sim$, where $(x,1) \sim (f(x),0)$ and $f : T^2 \to T^2$ is defined by identifying T^2 with $\mathbb{R}^2/\mathbb{Z}^2$, and setting $f(x,y) = (2x+y, -x)$.

2

Suppose that (C_*, d_C) and (D_*, d_D) are finitely generated free chain complexes defined over $\mathbb{Z}$, and that $f : C_* \to D_*$ is a chain map. Let $M_i = C_{i-1} \oplus D_i$, and define $d_f : M_i \to M_{i-1}$ by

$$d_f(x, y) = (d_C x, (-1)^i f(x) + d_D y)\,.$$

(a) Show that (M, d_f) is a chain complex, and that $H_*(M) = 0$ if and only if the map $f_* : H_*(C) \to H_*(D)$ is an isomorphism.

(b) Show that if $f_* : H_*(C \otimes \mathbb{Z}/p) \to H_*(D \otimes \mathbb{Z}/p)$ is an isomorphism for each prime p, then $H_*(C) \cong H_*(D)$.

3

Let $X = S^2 \times S^2$, and let $Y = X \times X$. Compute $H_*(Y)$. Give an explicit family of submanifolds S_i of Y with the property that $[S_i] \cdot [S_i] = 0$ for each i and $\{[S_i]\}$ is a basis for $H_4(Y)$.

Let $\Delta \subset Y$ be the set of points of the form (x, x) for $x \in X$. Express $[\Delta] \in H_4(Y)$ in terms of your basis. What is the self-intersection $[\Delta] \cdot [\Delta]$? (Give Y the orientation induced by fixing an orientation on X and taking the product orientation on $Y = X \times X$.)

4

(a) For which values of n is there a map $f : \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^{2n}$ for which the induced map $f_* : H_{2n}(\mathbb{RP}^n \times \mathbb{RP}^n; \mathbb{Z}/2) \to H_{2n}(\mathbb{RP}^{2n}; \mathbb{Z}/2)$ is an isomorphism?

(b) Let $f : S^2 \times S^2 \times S^2 \to \mathbb{CP}^3$ be a map of degree $d > 0$. What is the smallest possible value of d? Construct an example of a map with this degree.

5

Given $\gamma : S^{n-1} \to SO(k)$, explain how γ can be used to construct an oriented k-dimensional vector bundle V_γ over S^n. Construct an element $f_\gamma \in \pi_r(S^s)$ (for appropriate values of r and s) which vanishes if and only if V_γ has a non-vanishing section.

If $k > n$, show that V_γ can be decomposed as a direct sum $V_\gamma = V' \oplus \mathbb{R}$, where $\mathbb{R}$ is a trivial 1-dimensional bundle over S^n. (You may assume $\pi_r(S^s) = 0$ for $r < s$.) Show that the map $\pi_r(SO(s)) \to \pi_r(SO(s+1))$ induced by the inclusion is a surjection for $r < s$.

END OF PAPER