UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2010 1:30 pm to 4:30 pm

PAPER 16

ALGEBRAIC GEOMETRY

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Give the definitions of locally ringed spaces and morphisms between such spaces.

Let A and B be rings (commutative with identity element as always). Show that any ring homomorphism $A \longrightarrow B$ naturally induces a morphism

$$(Y = \text{Spec } B, \mathcal{O}_Y) \longrightarrow (X = \text{Spec } A, \mathcal{O}_X)$$

as locally ringed spaces. Conversely, show that any morphism

$$(Y = \text{Spec } B, \mathcal{O}_Y) \longrightarrow (X = \text{Spec } A, \mathcal{O}_X)$$

as locally ringed spaces is induced by a ring homomorphism $A \longrightarrow B$.

Moreover, show that $A \longrightarrow B$ is surjective if and only if the corresponding $f\colon Y \longrightarrow X$ is a homeomorphism onto a closed subset of X with $\mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ surjective.

[*You may assume without proof that* $(X = \text{Spec } A, \mathcal{O}_X)$ *and* $(Y = \text{Spec } B, \mathcal{O}_Y)$ *are locally ringed spaces.*]

2

Give the definition of a proper morphism of schemes, and state the valuative criterion of properness without proof. Let $f\colon X \longrightarrow Y$ and $g\colon Y \to Z$ be morphisms of Noetherian schemes. Show that

(i) f is proper if it is a finite morphism;

(ii) the composition gf is proper if both f and g are proper;

(iii) if gf is proper and g is separated, then f is proper;

(iv) if gf is proper, f is surjective, and g is separated and of finite type, then g is proper;

(v) if $X = \text{Spec } B$, $Y = \text{Spec } A$ and if f is proper, then f is a finite morphism. [*Here you may assume that* B *is a finitely generated* A*-algebra.*]

3

(i) Let X = Spec A, and let M and N be two A-modules. Show that $\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N}) \simeq \mathrm{Hom}_A(M, N)$.

(ii) Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. Give the proof of the following theorem: $\mathcal{F}$ is quasi-coherent if and only if for any open affine subscheme $U = \mathrm{Spec}\ A$ there is an A-module M such that $\mathcal{F}|_U \simeq \tilde{M}$.

(iii) Let X be a noetherian scheme and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{E} \to 0$ an exact sequence of $\mathcal{O}_X$-modules. Show that if $\mathcal{F}$ and $\mathcal{E}$ are quasi-coherent, then $\mathcal{G}$ is also quasi-coherent.

(iv) Give an example, with justification, of a scheme X and coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X such that $\mathcal{F}_x \simeq \mathcal{G}_x$ as $\mathcal{O}_x$-modules for every $x \in X$ but such that $\mathcal{F}$ and $\mathcal{G}$ are not isomorphic as $\mathcal{O}_X$-modules.

4

(i) Give the proof of the following theorem: if $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ a flasque $\mathcal{O}_X$-module, then $H^p(X, \mathcal{F}) = 0$ for any $p > 0$.

(ii) Let $X = \mathbb{A}^n_k$ where k is a field, and let $\mathcal{F}$ be the constant sheaf defined by the function field K of X. Compute the cohomology groups $H^p(X, \mathcal{F})$.

(iii) Let X be a reduced Noetherian scheme. Show that X is affine if and only if each irreducible component Y of X is affine. Here the scheme structure on Y is the closed subscheme structure induced by the scheme structure on X so that Y becomes an integral scheme, and near the generic point of Y, X and Y are isomorphic.

[*You may use the fact that for any morphism* $f \colon Z \longrightarrow X$ *of schemes and* $\mathcal{O}_X$*-module* $\mathcal{G}$*, there is a natural morphism* $\mathcal{G} \longrightarrow f_* f^* \mathcal{G}$.]

5

(i) Give an example, with justification, of a projective morphism $f\colon X \longrightarrow Y$ of schemes, and an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

of coherent sheaves on X such that the induced sequence

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{G} \longrightarrow f_*\mathcal{H} \longrightarrow 0$$

is not exact.

(ii) Let X be the closed subscheme of $\mathbb{P}^2_k = \operatorname{Proj}\, k[t_0, t_1, t_2]$ defined by the ideal of a homogeneous polynomial F of degree $d > 0$ where k is a field. Show that $\dim_k H^0(X, \mathcal{O}_X) = 1$ and $\dim_k H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2)$.

(iii) Give an example of a scheme X, an open affine covering $\mathcal{U} = (U_i)_{i \in I}$ and a sheaf $\mathcal{F}$ on X such that $\check{H}^1(\mathcal{U}, \mathcal{F}) \neq H^1(X, \mathcal{F})$, and justify your answer.

END OF PAPER