UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 28 May, 2010 1:30 pm to 4:30 pm

PAPER 15

DIFFERENTIAL GEOMETRY

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let X be a vector field on a manifold M. Define what it means for $c(t)$ to be an integral curve of X through a point $p \in M$, and what it means for ϕ_t to be the flow of X. If ϕ_t is the flow of X, what is the flow of the vector field $2X$?

(b) Suppose next that X and Y are two vector fields on M with flows ϕ_t and ψ_t respectively (which you may assume exist for sufficiently small t). Prove that ϕ_t and ψ_t commute if and only if $[X, Y] = 0$.

(c) Now consider the vector fields on $\mathbb{R}^2$ given by $X = y\frac{\partial}{\partial x}$ and $Y = \frac{x^2}{2}\frac{\partial}{\partial y}$. Show that $[X, Y]$ does not vanish. Find the flows of X and Y and verify directly that they do not commute. Determine if the set of vector fields whose flow exists for all time is preserved (i) under the Lie bracket and (ii) under addition of vector fields.

[*Hint: for (ii) you may find it helpful to consider the curve* $\gamma(t) = ((1 + \alpha t)^{-2}, \beta(1 + \alpha t)^{-3})$ *for suitable* α *and* β.]

2

Let M be a smooth manifold. Define what it means for $\pi \colon E \to M$ to be a smooth vector bundle on M, and what it means for ∇ to be a connection on E.

(a) Now let $\phi \colon M' \to M$ be a smooth map between smooth manifolds and $\pi \colon E \to M$ be a smooth vector bundle on M. Set

$$\phi^* E := \bigsqcup_{p \in M'} E_{\phi(p)} .$$

Show how $\phi^* E$ can be made into a smooth vector bundle on M' such that if s is a smooth section of E over an open set $U \subset M$ then $s \circ \phi$ is a smooth section of $\phi^* E$ over $\phi^{-1}(U)$.

(b) Given a connection ∇ on E prove that there is a unique connection ∇' on $\phi^* E$ with the property that

$$\nabla'_X (s \circ \phi) = \left(\nabla_{D\phi(X)} s\right) \circ \phi$$

for all vector fields X on M' and sections s of E over M.

(c) Define the curvature form of a connection. State and prove an equation that determines the curvature form of ∇' in terms of ∇ and ϕ.

3

(a) Let M be a connected manifold of dimension n. Define what it means for an atlas on M to be oriented, and show how the existence of a nowhere vanishing n-form gives rise to an oriented atlas.

(b) Now let g be a Riemannian metric on a connected oriented manifold M, which in local coordinates x_i is given by $g = \sum_{i,j=1}^{n} g_{ij}\, dx_i \otimes dx_j$. Show how

$$\omega_g := \sqrt{\det(g_{ij})}\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

gives a well-defined nowhere vanishing smooth n-form defined globally on all of M. [*Here* $\det(g_{ij})$ *denotes the determinant of the matrix valued function with entries* g_{ij}.]

(c) The divergence operator $\operatorname{div}\colon \operatorname{Vect}(M) \to C^\infty(M)$ is defined by the rule

$$d(\iota_X \omega_g) = (\operatorname{div} X)\omega_g \quad \text{for all } X \in \operatorname{Vect}(M)$$

where $\iota_X \omega_g$ is the $n-1$-form given by

$$(\iota_X \omega_g)(V_1, \ldots, V_{n-1}) = \omega_g(X, V_1, \ldots V_{n-1}) .$$

Prove that if M is a Riemannian manifold with boundary ∂M, and X is a vector field on M, then

$$\int_M (\operatorname{div} X)\omega_g = \int_{\partial M} g(X, N)\omega_{\tilde{g}}$$

where N is the outward unit normal to ∂M, and $\omega_{\tilde{g}}$ is the $(n-1)$-form associated to the induced Riemannian metric $\tilde{g}$ on ∂M.

4

(a) Let M be a manifold. Define what it means for a subset $Z \subset M$ to be an embedded submanifold of M.

(b) Suppose $F\colon M \to N$ is a smooth map between smooth manifolds, and $q \in N$ be such that $Z = F^{-1}(q)$ is non-empty. Assume that for all $p \in Z$ the map $DF|_p$ is surjective. Prove that Z is an embedded submanifold of M and determine its dimension. [*The inverse theorem for maps between subsets of* $\mathbb{R}^n$ *may be used without proof if stated clearly.*]

(c) Let $f\colon M \to N$ be a smooth map between smooth manifolds. Suppose that S is an embedded submanifold of N, such that $Z = f^{-1}(S)$ is non-empty and f is such that for all $p \in Z$

$$Df_p(T_pM) + T_{f(p)}S = T_{f(p)}N$$

(here the left hand side is the sum of vector spaces, which need not be a direct sum). Prove that $f^{-1}(S)$ is a submanifold of M. [*You may assume that if* $q \in S$ *then there is a chart* U *for* N *with* $q \in U$ *and coordinates* $x_1, \ldots, x_n$ *on* U *such that* $S \cap U = \{x_1 = \ldots = x_k = 0\}$.]

5

Let (M, g) be a Riemannian manifold and ∇ be a connection on M.

(a) (i) Define what it means for ∇ to be compatible with g.

(ii) Let $\gamma\colon [0,1] \to M$ be a smooth curve and $V(t)$ be a vector field along γ. Define what it means for $V(t)$ to be parallel along γ, and what it means for γ to be a geodesic.

(b) Show that ∇ is compatible with g if and only if $|V(t)|$ is constant with respect to t for any parallel vector field $V(t)$ along any smooth curve γ.

(c) Now suppose ∇ and ∇' are two connections on M, and define

$$A(X, Y) \,=\, \nabla_X Y - \nabla'_X Y \quad \text{for } X, Y \in \mathrm{Vect}(M)\,.$$

Show that ∇ and ∇' have the same geodesics if and only if $A(X, Y) = -A(Y, X)$ for all X, Y. Finally, assuming that ∇ is compatible with g, prove that ∇' is compatible with g if and only if

$$g(A(X, Y), Z) \,=\, -g(Y, A(X, Z)) \qquad \text{for all } X, Y, Z\,.$$

[*Any theorems used concerning the existence of parallel transport or existence of geodesics should be stated clearly.*]

END OF PAPER