UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2010 9:00 am to 12:00 pm

PAPER 14

THE X-RAY TRANSFORM IN GEOMETRY AND DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let ϕ_t be a smooth flow on a closed manifold. Define the *Anosov* property for ϕ_t. Give two different constructions of Anosov flows on 3-manifolds.

(b) Let F be the vector field of ϕ_t and $f : N \to \mathbb{R}$ any positive smooth function. Show that if ϕ_t is Anosov, then the flow of fF is also Anosov. [*If E^s is the stable bundle of ϕ_t, show that the stable bundle of the flow of fF will have the form*

$$\{v + \lambda(x, v)F(x) : \ v \in E^s(x)\}$$

for some continuous function λ depending linearly in v. Alternatively, you may use a result that characterises Anosov flows in terms of certain continuous quadratic forms, provided it is clearly stated.]

2

(a) State the Livsic Theorem for a transitive Anosov flow.

(b) Let M be a closed oriented surface with a Riemannian metric of negative curvature. Let SM be its unit sphere bundle, X the geodesic vector field on SM, and V the vertical vector field. Suppose β is a symmetric 2-tensor such that

$$\int_0^T \beta_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\, dt = 0$$

for every closed geodesic $\gamma : [0, T] \to M$. Show that there is a smooth function $u : SM \to \mathbb{R}$ such that $X(u)(x, v) = \beta_x(v, v)$ for all $(x, v) \in SM$. Moreover, if we let $\varphi := V^2(u) + u$ show that φ satisfies $VX(\varphi) = -2H(\varphi)$, where $H = [V, X]$.

(c) Assume the integral identity

$$\int_{SM} 2H(\varphi)VX(\varphi)\, d\mu = \int_{SM} (X(\varphi))^2\, d\mu + \int_{SM} (H(\varphi))^2\, d\mu - \int_{SM} K(V\varphi)^2\, d\mu.$$

where K is the Gaussian curvature, μ the Liouville measure and $u : SM \to \mathbb{R}$ any smooth function. Show that a symmetric 2-tensor β as in the previous part must be a potential 2-tensor, that is, there is a smooth vector field Z on M such that

$$\beta_x(v, v) = 2\langle \nabla_v Z(x), v\rangle.$$

3

(a) Define the *scattering relation* α of a compact simple Riemannian manifold with boundary $(M, \partial M, g)$.

(b) Let g_1 and g_2 be two simple metrics on M with the same boundary distance function. Show that there exists a diffeomorphism $\psi : M \to M$, which is the identity on the boundary, such that g_1 and $\psi^* g_2$ coincide on ∂M (that is, $g_1(u,v) = \psi^* g_2(u,v)$ for all $u, v \in T_x M$ and all $x \in \partial M$).

(c) Let g_1 and g_2 be two simple metrics such that they have the same boundary distance function and they agree on ∂M. Show g_1 and g_2 have the same scattering relation.

4

(a) Let N be a closed oriented 3-manifold and E a codimension one subbundle of TN of class C^2 which is transversally orientable and integrable. Define the *Godbillon-Vey invariant* $\mathrm{gv}(E)$ and show that your definition is independent of the choices made.

(b) Compute the Godbillon-Vey invariant of the weak stable bundle of the geodesic flow of a closed orientable surface of constant curvature -1.

5

(a) Let (M, g) be a closed oriented Riemannian surface with unit sphere bundle SM. Consider the flow ϕ_t on SM defined by the differential equation

$$\frac{D\dot{\gamma}}{dt} = f(\gamma(t))\, i\dot{\gamma},$$

where i indicates rotation by $\pi/2$ according to the orientation of the surface, $\gamma : \mathbb{R} \to M$ and $f : M \to \mathbb{R}$ is a given smooth function. Show that ϕ_t preserves the Liouville volume form of SM.

(b) Suppose that the flow ϕ_t in the previous part is Anosov and

$$\int_M f\, \Omega_a = 0,$$

where Ω_a is the area form. Show that ϕ_t preserves a smooth contact form if and only if f is identically zero. [*You may use results on the kernel of the X-ray transform provided they are clearly stated.*]

END OF PAPER