UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 2 June, 2010 1:30 pm to 4:30 pm

PAPER 13

COMPLEX MANIFOLDS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Given holomorphic functions $f_1, \ldots, f_n$ on a polydisc $\Delta^n \subset \mathbb{C}^n$ centred at a point $\mathbf{a} = (a_1, \ldots, a_n)$ with $\partial f_i/\partial z_j = \partial f_j/\partial z_i$ for all i, j, show that there exists a holomorphic function f on Δ^n such that $\partial f/\partial z_i = f_i - f_i(\mathbf{a})$ for all i. [*Hint: Define $f(z_1, \ldots, z_n)$ as a certain sum of n integrals; standard facts from Analysis may be assumed.*]

Let M be a complex manifold of dimension n. If Ω^1_M denotes the sheaf of holomorphic 1-forms, and $\tilde{\Omega}^1_M$ denotes the subsheaf of *closed* holomorphic 1-forms, deduce that there is a short exact sequence of sheaves

$$0 \to \mathbb{C} \hookrightarrow \mathcal{O}_M \xrightarrow{\partial} \tilde{\Omega}^1_M \to 0\,.$$

If M is now assumed to be compact, show that any global complex-valued function f with $\partial\bar{\partial} f = 0$ must be constant. [*You may assume that for any non-constant harmonic function g on a domain in $\mathbb{C}$, there are no local maxima for $|g|$.*]

Still assuming that M is compact, suppose that θ is a non-zero smooth $(n,0)$-form on M; explain why the integral over M of the (n,n)-form $\theta \wedge \bar{\theta}$ is non-zero. If ψ denotes a holomorphic $(n-1)$-form on M, deduce that ψ is closed.

Suppose now that $n = 2$, so that M is a compact complex surface. Show that there is an inclusion $H^0(M, \Omega^1_M) \hookrightarrow H^1_{\mathrm{DR}}(M, \mathbb{C})$ into the first de Rham cohomology group. Prove that

$$H^0(M, \Omega^1_M) \cap \overline{H^0(M, \Omega^1_M)} = 0$$

in $H^1_{\mathrm{DR}}(M, \mathbb{C})$, and hence that $2h^{1,0} \leqslant b_1$, where $b_1 = \dim_{\mathbb{C}} H^1_{\mathrm{DR}}(M, \mathbb{C})$. From the above short exact sequence of sheaves, prove furthermore that $b_1 \leqslant h^{1,0} + h^{0,1}$, and hence that $h^{1,0} \leqslant h^{0,1}$. Give an example of a compact complex surface for which this last inequality is strict.

2

Prove Cartan's equation relating the curvature and connection matrices (with respect to some local frame) of a connection on a smooth complex vector bundle, namely

$$\Theta \, = \, d\theta - \theta \wedge \theta \, . \tag{†}$$

If E is a hermitian holomorphic vector bundle of rank r over a complex manifold M, describe the defining properties for the Chern connection on E, and prove that there exists a unique connection with these properties. With the curvature considered as $\Theta \in A^{1,1}(\mathrm{End}(E)) = A^{0,1}(\Omega^1(\mathrm{End}(E)))$, where $\mathrm{End}(E) = \mathrm{Hom}(E,E)$ is the endomorphism bundle, show that $\bar\partial\,\Theta = 0$, and hence that Θ determines a class Ψ in the Dolbeault cohomology group $H^1(M, \Omega^1(\mathrm{End}(E))$.

Suppose now that we have local holomorphic frames $e_1^{(\alpha)}, \ldots, e_r^{(\alpha)}$ for E over U_α, with $\mathcal{U} = \{U_\alpha\}$ an open cover of M. Suppose the transition functions of E with respect to $\mathcal{U}$ are represented by the transpose of matrices $g_{\alpha\beta}$; that is the frames transform via the relation

$$e_i^{(\beta)} = \sum_j (g_{\alpha\beta})_{ij}\, e_j^{(\alpha)} \, .$$

Show that Ψ corresponds to a Čech cohomology class $(\sigma_{\alpha\beta}) \in H^1(\mathcal{U}, \Omega^1(\mathrm{End}(E))$, where $\sigma_{\alpha\beta}$ is the section of $\Omega^1(\mathrm{End}(E))$ over $U_\alpha \cap U_\beta$ represented with respect to the local holomorphic frame $e_1^{(\beta)}, \ldots, e_r^{(\beta)}$ by the matrix of holomorphic 1-forms $(\partial g_{\alpha\beta})\, g_{\alpha\beta}^{-1}$. Hence show that Ψ depends on neither the choice of hermitian metric nor local trivialization for E. Show also that there is a corresponding well-defined class

$$\Psi^{(k)} \, \in \, H^k(M, \Omega^k(\mathrm{End}(E)) \, ,$$

and hence by contraction a class

$$\mathrm{tr}(\Psi^{(k)}) \, \in \, H^k(M, \Omega^k) \, ,$$

for all $k > 0$. When M is compact and Kähler, explain why $(\frac{i}{2\pi})^k\,\mathrm{tr}(\Psi^{(k)})$ determines a real class in $H^{2k}(M, \mathbb{C})$.

3

State the defining property of the Hodge $*$-operator, $*: \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{n-p,n-q}$, on the sheaf of (p,q)-forms on an n-dimensional complex manifold M equipped with a hermitian metric. For M compact, explain briefly how this determines a hermitian inner-product on the global (p,q)-forms $A^{p,q}(M)$. State carefully the Hodge theorem concerning the decomposition of $A^{p,q}(M)$ by means of the $\bar\partial$-Laplacian $\Delta_{\bar\partial}$, and deduce the standard orthogonal decomposition

$$A^{p,q}(M) = \mathcal{H}_{\bar\partial}^{p,q} \oplus \bar\partial A^{p,q-1} \oplus \bar\partial^* A^{p,q+1},$$

with the $\bar\partial$-closed forms being the sum of the first two factors. [*Standard properties of* $\bar\partial^* = - * \bar\partial *$ *may be assumed.*]

Suppose now that M is also Kähler; show that $\partial\bar\partial^* + \bar\partial^*\partial = 0$. [*You may assume the result expressing* $\bar\partial^*$ *as a certain commutator of operators, provided you state it precisely.*] Define the Laplacians Δ_d and Δ_∂, and prove that $\Delta_d = \Delta_\partial + \Delta_{\bar\partial}$ and $\Delta_\partial = \Delta_{\bar\partial}$.

Let η be a $\bar\partial$-exact (p,q)-form ($p \geqslant 1$, $q \geqslant 1$) on a compact Kähler manifold M; show that $\eta = \bar\partial\bar\partial^*\alpha$ for some form $\alpha \in A^{p,q}(M)$. If η is also ∂-closed, prove that $\partial\bar\partial^*\alpha$ is harmonic; by considering the orthogonal decomposition corresponding to Δ_∂, deduce that $\bar\partial^*\alpha$ is ∂-closed. Conclude that such an η can be expressed as $\eta = \partial\bar\partial\phi$ for some $\phi \in A^{p-1,q-1}(M)$.

Given two Kähler forms ω_1 and ω_2 on M which define the same de Rham cohomology class, show that there is a smooth real-valued function f with $\omega_2 = \omega_1 + i\partial\bar\partial f$.

4

What is meant by the *canonical line bundle* K_M of a complex manifold M? Let $V \subset M$ be an n-dimensional complex submanifold of an m-dimensional complex manifold M; define what is meant by the *normal bundle* $N_{V/M}$. State the *adjunction formula* relating K_V and K_M. In the case when V is of codimension one, define the line bundle $[V]$ on M. State (without proof) the relation between $[V]$ and $N_{V/M}$.

Suppose now E is a holomorphic hermitian vector bundle of rank r on a complex manifold V, and $F \subset E$ is a holomorphic subbundle of rank s, equipped with the induced hermitian structure. Let $\pi : E \to F$ be the orthogonal projection map (a smooth map of the holomorphic bundles) and D_E denote the Chern connection on E. Show that the Chern connection D_F of F is given by the relation $D_F = \pi \circ D_E$. Choose a local unitary frame $e_1, \ldots, e_s$ for F and extend to a local unitary frame $e_1, \ldots, e_r$ for E. With respect to this unitary frame, show that D_E has connection matrix

$$\begin{pmatrix} \theta_1 & \bar{A}^t \\ -A & \theta_2 \end{pmatrix}$$

where θ_1 is the connection matrix for D_F with respect to $e_1, \ldots, e_s$, and θ_2 and A are matrices of 1-forms. Assuming Cartan's equation (†) from Question 2 above, find an expression for the curvature matrix Θ_F for F in terms of the curvature matrix Θ_E and A.

Let $V \subset M$ now be a codimension one submanifold of a complex torus M and L is a positive line bundle on M. Show that $(L \otimes [V]^{\otimes a})|_V$ is positive for all $a \geqslant 0$. Assuming the Kodaira Vanishing Theorem, show that $H^i(M, L \otimes [V]^{\otimes a}) = 0$ for all $i > 0$ and $a \geqslant 0$.

END OF PAPER