UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2010 9:00 am to 11:00 am

PAPER 12

PROBABILISTIC COMBINATORICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Let $G^{(r)}_{\mathbb{N},p}$ be the random r-uniform hypergraph with vertex set $\mathbb{N}$ in which every r-set is chosen with probability p, where $0 < p < 1$. Show that a.s. $G^{(r)}_{\mathbb{N},p}$ is isomorphic to a fixed r-uniform hypergraph $G^{(r)}_{\text{univ}}$.

(ii) Delete the maximal vertex of each hyperedge E of $G^{(3)}_{\text{univ}}$ from E to obtain a *graph* G_0. Show that G_0 is isomorphic to $G^{(2)}_{\text{univ}}$.

2

(i) Prove a concentration of probability inequality for a Lipschitz function on the symmetric group S_n, considered as a probability space, with Hamming distance.

(ii) Let A and B be non-empty subsets of S_n, and write $d(A,B)$ for their Hamming distance. Show that

$$\min\{|A|,\, |B|\} \leqslant n!\, e^{-d^2/8n}\,.$$

[The results you use should be clearly stated.]

3

Let $z = (z_1, \ldots, z_n)$ be a sequence of n points chosen at random in the right-angled triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$, and let $L_n = L_n(z)$ be the maximal number of points z_i forming a convex polygon with $(0,0)$ and $(1,1)$. [Thus $L_n(z)$ is the maximal ℓ such that the points $(0,0)$, $z_{i_1}, z_{i_2}, \ldots, z_{i_\ell}$ and $(1,1)$ are the vertices of a convex $(\ell+2)$-gon.]

(i) Sketch a proof of the fact that the median of the random variable L_n is at most $cn^{1/3}$ for some $c > 0$. [You may find it helpful to consider the triangles formed by the tangents of the hyperbola $y = x^2$ at the points $(k/n^{1/3}, k^2/n^{2/3})$.]

(ii) Let $1 \leqslant b < a$ be positive integers, and set $A = \{z : \; L_n(z) \geqslant a\}$ and $B = \{z : \; L_n(z) \leqslant b\}$. Show that

$$d_T(A,B) \geqslant \frac{a-b}{\sqrt{a}}\,,$$

where $d_T(A,B)$ is the Talagrand distance of the sets A and B.

(iii) Let $\omega(n) \to \infty$. Prove that whp L_n is concentrated in an interval of length $\omega(n)n^{1/6}$.

4

Let m, n, r and s be positive integers with $rm = sn$, and let G be an (r, s)-regular bipartite graph. Thus $V(G)$ is the disjoint union of two sets, U and W, with $|U| = m$, $|W| = n$, every edge of G joins a vertex of U to a vertex of W, every vertex of U has degree r and every vertex of W has degree s.

(i) Prove that G has at most $(2^r + 2^s - 1)^{m/s}$ independent sets of vertices.

(ii) Show that if $s|m$ then equality holds for some graph G.

[All results you quote should be stated precisely.]

END OF PAPER