UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Tuesday, 1 June, 2010 9:00 am to 12:00 pm

## PAPER 10

## COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

State and prove the Kruskal-Katona theorem.

Show that a graph with 15 edges may contain 15 copies of $K_4$ but no more.

**2**

Let $\mathcal{A}_i = \{A \in [n]^{(r)} : |A \cap [t+2i]| \geqslant t+i\}$ and $n_k = (r-t+1)(2+(t-1)/k)$. By comparing $\mathcal{A}_{i+1} \setminus \mathcal{A}_i$ with $\mathcal{A}_i \setminus \mathcal{A}_{i+1}$, or otherwise, show that if $n_{k+1} < n < n_k$ then $\max_i |\mathcal{A}_i| = |\mathcal{A}_k|$.

State and prove the Ahlswede-Khachatrian theorem giving the value of $M(n,r,t)$, the maximum size of a $t$-intersecting family $\mathcal{A} \subset [n]^{(r)}$.

[*You may use lemmas about compressions preserving $t$-intersections, and the existence of a generating family on a small ground-set, provided you state them clearly.*]

**3**

What does it mean that an $r$-uniform hypergraph is *strongly* $(r+t)$*-saturated*?

Show that a strongly $(r+t)$-saturated $r$-uniform hypergraph of order $n$ has at least $\binom{n}{r} - \binom{n-t}{r}$ edges.

Let $(R_i, S_i)$, $i \in I$ be a family of pairs of subsets with $R_i \in [n]^{(r)}$ and $S_i \in [n]^{(s)}$, such that $R_i \cap S_j \neq \emptyset$ if and only if $i = j$. Show that if $|I| \geqslant 2$ then $|I| \leqslant n-r-s+2$ and that equality can be attained.

[*Hint: pick* $x_i \in R_i \cap S_i$.]

**4**

State and prove the Sauer-Shelah lemma on families that shatter $k$-sets.

Let $\mathcal{A}$ be a collection of finite subsets of some infinite set $X$ and, for $Y \subset X$, let $\mathcal{A}_{|Y} = \{Y \cap A : A \in \mathcal{A}\}$. Suppose that, for every $k \geqslant 1$, there is a set $Y \in X^{(k)}$ with $|\mathcal{A}_{|Y}| \geqslant 2^{k-1}$. Show that, for every $k \geqslant 1$, there is a set $Z \in X^{(k)}$ that is shattered by $\mathcal{A}$.

Must there be an infinite $Z \subset X$ such that $\mathcal{A}_{|Z}$ contains every finite subset of $Z$?

## END OF PAPER