UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 1 June, 2010 9:00 am to 12:00 pm

PAPER 10

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the Kruskal-Katona theorem.

Show that a graph with 15 edges may contain 15 copies of K_4 but no more.

2

Let $\mathcal{A}_i = \{A \in [n]^{(r)} : |A \cap [t+2i]| \geqslant t+i\}$ and $n_k = (r-t+1)(2+(t-1)/k)$. By comparing $\mathcal{A}_{i+1} \setminus \mathcal{A}_i$ with $\mathcal{A}_i \setminus \mathcal{A}_{i+1}$, or otherwise, show that if $n_{k+1} < n < n_k$ then $\max_i |\mathcal{A}_i| = |\mathcal{A}_k|$.

State and prove the Ahlswede-Khachatrian theorem giving the value of $M(n,r,t)$, the maximum size of a t-intersecting family $\mathcal{A} \subset [n]^{(r)}$.

[*You may use lemmas about compressions preserving t-intersections, and the existence of a generating family on a small ground-set, provided you state them clearly.*]

3

What does it mean that an r-uniform hypergraph is *strongly* $(r+t)$*-saturated*?

Show that a strongly $(r+t)$-saturated r-uniform hypergraph of order n has at least $\binom{n}{r} - \binom{n-t}{r}$ edges.

Let (R_i, S_i), $i \in I$ be a family of pairs of subsets with $R_i \in [n]^{(r)}$ and $S_i \in [n]^{(s)}$, such that $R_i \cap S_j \neq \emptyset$ if and only if $i = j$. Show that if $|I| \geqslant 2$ then $|I| \leqslant n - r - s + 2$ and that equality can be attained.

[*Hint: pick* $x_i \in R_i \cap S_i$.]

4

State and prove the Sauer-Shelah lemma on families that shatter k-sets.

Let $\mathcal{A}$ be a collection of finite subsets of some infinite set X and, for $Y \subset X$, let $\mathcal{A}_{|Y} = \{Y \cap A : A \in \mathcal{A}\}$. Suppose that, for every $k \geqslant 1$, there is a set $Y \in X^{(k)}$ with $|\mathcal{A}_{|Y}| \geqslant 2^{k-1}$. Show that, for every $k \geqslant 1$, there is a set $Z \in X^{(k)}$ that is shattered by $\mathcal{A}$.

Must there be an infinite $Z \subset X$ such that $\mathcal{A}_{|Z}$ contains every finite subset of Z?

END OF PAPER