UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 27 May, 2010 1:30 pm to 4:30 pm

PAPER 1

LIE GROUPS, LIE ALGEBRAS, AND THEIR REPRESENTATIONS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define a *derivation* of a Lie algebra $\mathfrak{g}$ over a field k to be a k-linear map $f : \mathfrak{g} \to \mathfrak{g}$ such that $f([x,y]) = [f(x),y] + [x,f(y)]$ for all x,y in $\mathfrak{g}$. Show that the derivations $\mathrm{Der}(\mathfrak{g})$ form a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. Show that

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto \mathrm{ad}\, x \end{aligned}$$

is a homomorphism of Lie algebras. Show that the image $\mathrm{ad}(\mathfrak{g})$ forms an ideal in $\mathrm{Der}(\mathfrak{g})$ (called the ideal of "inner derivations").

2

Let R be an irreducible root system in a real vector space E. Show that E is irreducible as a representation of the Weyl group.

3

Find the dimensions of all weight spaces for the representation $V \otimes \Lambda^2 V$ of the complex Lie algebra $\mathfrak{sl}(n)$. Here $V = \mathbf{C}^n$ denotes the standard representation, and we assume that $n \geqslant 3$. Find the highest weights of all the irreducible summands of $V \otimes \Lambda^2 V$, as a representation of $\mathfrak{sl}(n)$.

4

Let

$$\mathfrak{g} = \mathfrak{sp}(2n) := \{A \in M_{2n}(\mathbf{C}) : AJ + JA^t = 0\},$$

where J is the $2n \times 2n$ matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

Let $\mathfrak{t}$ be the space of diagonal matrices in $\mathfrak{g}$, which is a Cartan subalgebra in $\mathfrak{g}$.

(a) By decomposing $\mathfrak{g}$ as a $\mathfrak{t}$-module, write down the set R of roots of $\mathfrak{g}$. Choose a set R^+ of positive roots. Write down the set Δ of simple roots, the highest root θ, and ρ (in the notation of the Weyl character formula).

Draw the Dynkin diagram of $\mathfrak{g}$, and label it by the simple roots.

(b) Show that $\mathfrak{so}(5) \cong \mathfrak{sp}(4)$.

(c) For each root $\alpha \in R$, write the reflection $s_\alpha : \mathfrak{t} \to \mathfrak{t}$ explicitly. Determine the Weyl group W, proving your answer.

5

Let H be a closed subgroup of a Lie group G. That is, H is a closed subset of G which is also a subgroup. This problem will show that H is automatically a closed Lie subgroup of G.

(a) Show that either H is discrete or H contains a nontrivial one-parameter subgroup of G.

(b) Let $\mathfrak{h}$ be the set of elements x in the Lie algebra $\mathfrak{g}$ of G such that H contains the one-parameter subgroup of G in the direction x. Show that $\mathfrak{h}$ is an $\mathbf{R}$-linear subspace of $\mathfrak{g}$. More strongly, show that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

(c) Show that H is a closed Lie subgroup of G with Lie algebra $\mathfrak{h}$. [*Hint: The main point is to show that H is a smooth submanifold of G.*]

END OF PAPER