UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2009 1:30 pm to 4:30 pm

PAPER 9

METHODS IN ANALYSIS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 a) Let $2 \leqslant p < \infty$ and $n \in \mathbb{N}$. Making use of the inequality

$$\|f+g\|_p^p + \|f-g\|_p^p \leqslant 2^{p-1}\left(\|f\|_p^p + \|g\|_p^p\right),$$

valid for all $f, g \in L^p(\mathbb{R}^n)$, prove the following statement.

Let $K \subset L^p(\mathbb{R}^n)$ be closed and convex, and let $f \notin K$. Then there exists $h \in K$ such that

$$\inf_{g \in K} \|g - f\|_p = \|h - f\|_p\,. \tag{1}$$

b) Prove that if h satisfies (**??**), then

$$\mathrm{Re} \int |h(x) - f(x)|^{p-2}\,(\overline{h(x)} - \overline{f(x)})\,(h(x) - g(x))dx \leqslant 0 \qquad \text{for all } g \in K\,.$$

[*You can make use of the fact that, if one defines* $N(t) = \|f + tg\|_p^p$ *for* $f, g \in L^p(\mathbb{R}^n)$, $f \neq 0$, *and* $t \in \mathbb{R}$, *then* $N(t)$ *is differentiable in* t *and*

$$\frac{d}{dt}N(t)|_{t=0} = 2Re \int |f(x)|^{p-2}\,\overline{f(x)}\,g(x).$$

]

c) Let $2 \leqslant p < \infty$ and $1 < q \leqslant 2$ be such that $1/p + 1/q = 1$. Using the results from part a) and b), show that the map $\phi : L^q(\mathbb{R}^n) \to (L^p(\mathbb{R}^n))^*$ defined by $\phi(u) \equiv \phi_u$ for all $u \in L^q(\mathbb{R}^n)$ and $\phi_u(v) = \int u(x)v(x)\,dx$ for all $v \in L^p(\mathbb{R}^n)$ defines an isometric isomorphism. [*You have to prove that the map is well defined, that it is linear, that* $\|\phi_u\|_{(L^p)^*} = \|u\|_{L^q}$ *for every* $u \in L^q(\mathbb{R}^n)$ *and that* ϕ *is surjective. You can make use of the Hölder inequality without proving it.*]

2 Suppose (Ω, Σ, μ) is a measure space.

a) For $1 \leqslant p \leqslant \infty$, prove the completeness of $L^p(\Omega, d\mu)$. [*You may use the fact that* $\|.\|_p$ *is a norm as well as the monotone and the dominated convergence theorems without proof.*]

b) Suppose that $1 \leqslant p < q < r \leqslant \infty$. Prove that, for every $C_p < \infty$, $C_r < \infty$, and $C_q > 0$, there exist $\epsilon > 0$ and $M > 0$ such that $\mu(\{x : |f(x)| > \epsilon\}) > M$ for all $f \in L^p(\Omega, d\mu) \cap L^r(\Omega, d\mu)$ such that $\|f\|_p \leqslant C_p$, $\|f\|_q \geqslant C_q$, and $\|f\|_r \leqslant C_r$.

c) Find examples to show that the conclusion of b) does not necessarily hold true if the condition $\|f\|_p \leqslant C_p$ or the condition $\|f\|_r \leqslant C_r$ is removed. [*Hint: you can take* $\Omega = \mathbb{R}$ *and* μ *to be Lebesgue's measure.*]

3 a) Assume that $n \geqslant 3$. Suppose that $f, f_j \in H^1(\mathbb{R}^n)$ for every $j \in \mathbb{N}$ are such that $f_j \to f$ weakly in $H^1(\mathbb{R}^n)$ as $j \to \infty$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded set and that χ_Ω denotes the characteristic function of Ω. Show that $\chi_\Omega f_j \to \chi_\Omega f$ strongly in $L^q(\mathbb{R}^n)$, for all $q < 2n/(n-2)$.

b) State and prove the Poincaré inequality for $f \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded, connected, open set having the cone-property, and where $p < n$. [*In the proof you can make use of the Rellich-Kondrashov Theorem on general sets having the cone-property.*]

4 a) Suppose that $\Omega \subset \mathbb{R}^n$ is open. Suppose that $T_n \in \mathcal{D}'(\Omega)$ for all $n \in \mathbb{N}$, and $T \in \mathcal{D}'(\Omega)$. What does $T_n \to T$ in $\mathcal{D}'(\Omega)$ mean? How is the distribution $\partial_{x_j} T$ defined? Show that the derivative of a distribution is a distribution. Show that $T_n \to T$ implies that $\partial_{x_j} T_n \to \partial_{x_j} T$ in $\mathcal{D}'(\Omega)$.

b) Explain the meaning of the equation

$$-\Delta \frac{1}{|x|} = 4\pi\delta \qquad \text{in } \mathcal{D}'(\mathbb{R}^3)\,. \tag{1}$$

Prove (**??**).

c) Consider the sequence of functions on $\mathbb{R}^2$:

$$g_n(x) := \begin{cases} c_n(1-|x|^4)^n & \text{if } |x| \leqslant 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

with

$$c_n^{-1} = \int_{|x|\leqslant 1} dxdy\,(1-|x|^4)^n$$

Prove that $g_n \to \delta$ in $\mathcal{D}'(\mathbb{R}^2)$. [*Hint: it may be useful to start by finding an upper bound for the constants c_n.*]

5 a) Suppose that $\Omega \subset \mathbb{R}^n$ is open, and let $f \in L^1_{\mathrm{loc}}(\Omega)$ be a real valued function. What does it mean for f to be *subharmonic* on Ω? What does it mean for f to be *superharmonic* on Ω? What does it mean for f to be *harmonic* on Ω?

b) Show that, if f_n is a sequence of subharmonic functions on Ω, then $g(x) = \sup_{n \geqslant 1} f_n(x)$ is also subharmonic.

If $f \in L^1_{\mathrm{loc}}(\Omega)$ is subharmonic, we proved in class that there exists a unique function $\widetilde{f} : \Omega \to \mathbb{R} \cup \{-\infty\}$ such that $f(x) = \widetilde{f}(x)$ for a.e. $x \in \Omega$, $\widetilde{f}$ is upper semicontinuous, and $\widetilde{f}$ is subharmonic for all $x \in \Omega$. This function then satisfies the mean-value inequality for all $x \in \Omega$. You can make use of these facts to answer the two following questions.

c) Show that if $f \in L^1_{\mathrm{loc}}(\Omega)$ is harmonic on Ω and $f = \widetilde{f}$, then $f \in C^\infty(\Omega)$.

d) State and prove the strong maximum principle.

END OF PAPER