UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2009 9:00 am to 11:00 am

PAPER 81

THE FLUID DYNAMICS OF SWIMMING ORGANISMS

*Attempt no more than **TWO** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

*This is an **OPEN BOOK** examination.*
Candidates may bring handwritten notes and lecture handouts into the examination.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 An infinitely thin 'fish' has uniform depth and density, a rigid body of length αL and a rigid tail of length L which is hinged to the body and can rotate relative to it. The mass of the fish is M, its moment of inertia about a vertical axis through the centre of mass is I, and the added mass per unit length is m. The lateral displacement, relative to a straight line through the body, is

$$\begin{aligned} h_0(x,t) &= 0 \qquad\qquad -\alpha L < x < 0 \\ &= h_1 \frac{x}{L} f(t) \qquad 0 < x < L \end{aligned}$$

where h_1 is the amplitude at the tip of the tail, $x = L$.

Taking into account the recoil correction, use Lighthill's small-amplitude elongated-body theory to show that, for $\alpha \gg 1$, the lateral displacement $R(t)$ of the centre of mass and the recoil rotation about it, $\theta(t)$, are given approximately by

$$\ddot{R}(M'-1) + L\ddot{\theta} - 2U\dot{\theta} = \tfrac{1}{2}\alpha^{-1} h_1 \left(\ddot{f} + \frac{4U\dot{f}}{L} \right)$$

and

$$-\ddot{R}(M'-1) + L\ddot{\theta}2\alpha\big(I' + \tfrac{1}{6}\big) + 2U\dot{\theta} = \alpha^{-2} h_1 \left(\frac{2}{3}\ddot{f} + \frac{2U}{L}\dot{f} \right),$$

where U is the swimming speed, assumed constant, $M' = M/m(\alpha L)$ and $I' = I/m(\alpha L)^3$.

In the case $f = \cos \omega t$, deduce that the mean thrust is approximately equal to

$$\frac{m}{4}\left[h_1^2\omega^2 - U^2 + \frac{h_1^2\omega^2}{\alpha} \left(\frac{1}{M'-1} + \frac{1}{4\left(I' + \frac{1}{6}\right)} \right) \right].$$

Comment on this result.

2 A spermatozoon swims in a fluid of viscosity μ with instantaneous head velocity $[-U(t), -V(t)]$ in (x, y) coordinates, by passing a planar wave along its inextensible flagellum, whose length is L. If it were not swimming the flagellum would be aligned with the x-axis. The y-coordinate of a material point at distance s along the centreline of the flagellum, measured from the point O where it is attached to the head, which is spherical, and in a frame of reference moving with O, is

$$Y(s,t) = H\mathrm{e}^{ks}\cos[k(s-ct)]\,,$$

where H, k and c are constants.

Ignoring the rotation of the spermatozoon, use resistive-force theory, for small kH, to find the instantaneous value of $V(t)$ and to show that the mean swimming speed is

$$\overline{U} \approx \frac{ck^2H^2}{4kL}\frac{(1-\gamma)}{(\delta+\gamma)}\left\{\mathrm{e}^{2kL} - 1 - \frac{1}{(1+\delta)kL}\left(\mathrm{e}^{2kL} - 2\,\mathrm{e}^{kL}\cos kL + 1\right)\right\}\,,$$

where $\gamma = K_T/K_N$, the ratio of the tangential and normal resistance coefficients, and the drag on the head is equal to $\mu K_N L\delta(U, V)$.

Calculate, to leading order in kH, the time-dependent bending moment that is exerted by the contractile apparatus at $s = 0$. [*There is no need to simplify the expression you obtain.*]

3 A dilute suspension of identical, spherical, bottom-heavy, swimming micro-organisms occupies a long circular cylinder of radius R which is rotating steadily at angular velocity Ω about its axis, which is horizontal and in the x_2-direction; x_3 is vertical and B is the gyrotactic reorientation time. Bioconvection does not occur.

(a) Show that, if randomness in cell swimming is neglected and $B\Omega < 1$, individual cells follow circular paths, centred on an axis A that is displaced by a distance V_s/Ω from the cylinder axis O, where V_s is the cell swimming speed, and derive the angle α that the plane containing the two axes A and O makes with the horizontal.

(b) In the case where randomness in the cell swimming direction $\mathbf{p}$ cannot be neglected, and for $B\Omega \gg 1$, solve the Fokker-Planck equation for the probability density function $f(\mathbf{p})$ in the form

$$f = f_0 + \frac{\alpha}{B\Omega}\sin\theta\cos\phi + O\left[(B\Omega)^{-2}\right],$$

where (θ, ϕ) are the polar coordinates of $\mathbf{p}$ with $\theta = 0$ in the x_3-direction and $\phi = 0$ in the (x_1, x_3)-plane, and α is a constant to be evaluated. You may assume that f_0 is isotropic and that $\lambda = 1/BD_R = O(1)$, where D_R is the rotational diffusivity.

Calculate the mean swimming direction $< \mathbf{p} >$ to leading order.

(c) Under the conditions of part (b), show that the steady-state cell-conservation equation reduces to the following dimensionless form in plane polar coordinates (Rr, θ') (where $\theta' = 0$ in the x_1-direction):

$$\frac{\partial^2 n}{\partial r^2} + \frac{1}{r}\frac{\partial n}{\partial r} + \frac{1}{r^2}\frac{\partial^2 n}{\partial \theta'^2} - \beta^2\frac{\partial n}{\partial \theta'} + \epsilon\left(\cos\theta'\frac{\partial n}{\partial r} - \frac{\sin\theta'}{r}\frac{\partial n}{\partial \theta'}\right) = 0,$$

where $N_0 n$ is the number of cells per unit volume (average value N_0),

$$\beta^2 = \frac{\Omega R^2}{D}, \quad \epsilon = \frac{2V_s R}{3B\Omega D},$$

and D is the translational diffusivity, assumed isotropic. Show too that the boundary condition on $r = 1$ is

$$\frac{\partial n}{\partial r} + \epsilon\, n\cos\theta' = 0.$$

Assuming that $\epsilon \ll 1$ and $\beta \gg 1$, show that the steady-state cell distribution is given approximately by the real part of

$$n = 1 - \frac{\epsilon}{\beta}\,\mathrm{e}^{i(\theta' - \pi/4)}\exp\left[-\,\mathrm{e}^{i\pi/4}\,\beta(1-r)\right].$$

4 Consider a dilute suspension of identical, spherical, bottom-heavy, swimming micro-organisms in fluid of density ρ and kinematic viscosity ν. An individual cell has swimming velocity $V_s\mathbf{p}$, where $\mathbf{p}$ is a unit vector, and the centre of mass is located at position $-h\mathbf{p}$ relative to the centre of the sphere. The cell has volume v, mass m, and density $\rho + \Delta\rho = m/v$. Neglecting randomness of cell behaviour, explain why $\mathbf{p}$ is given by

$$\frac{1}{B}\left[\widehat{\mathbf{k}} - (\widehat{\mathbf{k}}\cdot\mathbf{p})\mathbf{p}\right] = \frac{1}{2}\widehat{\boldsymbol{\omega}} \wedge \mathbf{p}\,, \tag{1}$$

where $\widehat{\mathbf{k}}$ is the unit vector directed vertically upwards, $\widehat{\boldsymbol{\omega}}$ is the vorticity in the ambient flow and

$$B = \rho\nu\alpha_{\perp}v/mgh\,,$$

where $\alpha_{\perp}$ is a dimensionless constant. What is the physical interpretation of B?

The suspension occupies a chamber $-H \leqslant \widehat{z} \leqslant 0$, of horizontal extent $\gg H$, and exhibits bioconvection when the average cell volume fraction $\widehat{n}_0$ exceeds a critical value which we seek to calculate.

(a) The basic state is one in which the fluid velocity $\widehat{\mathbf{u}} = \mathbf{0}$ and all the cells are taken to be swimming upwards. Random behaviour is modelled by an isotropic translational diffusivity D. Use the cell-conservation equation to show that the cell volume fraction $\widehat{n}$ is given by

$$\widehat{n} = \widehat{n}_0\,\frac{h\exp[h\widehat{z}/H]}{1-\mathrm{e}^{-h}} \equiv \widehat{n}_0\,\overline{n}(z)\,,$$

where $h = V_sH/D$ and $z = \widehat{z}/H$.

(b) Write down the differential equations governing small perturbations about the basic state, assuming that the stress tensor is Newtonian, and non-dimensionalise them using the following variables:

$$(x,z) = \frac{(\widehat{x},\widehat{z})}{H}, \quad t = \frac{\nu\widehat{t}}{H^2}, \quad n = \frac{\widehat{n}}{\widehat{n}_0}, \quad \mathbf{u} = (u,v,w) = \frac{H\widehat{\mathbf{u}}}{D}, \quad P_e = \frac{H^2\widehat{P}_e}{\rho\nu D}$$

where $\widehat{P}_e$ is the pressure perturbation. Show that, for disturbances in which

$$[n - \overline{n}(z), w] = [N(z), W(z)]\exp[\sigma t + i\kappa x]\,,$$

the equations reduce to

$$\left(\frac{d^2}{dz^2} - \kappa^2 - \sigma\right)\left(\frac{d^2}{dz^2} - \kappa^2\right)W = -R\kappa^2 N$$

$$\left(\frac{d^2}{dz^2} - h\frac{d}{dz} - \kappa^2 - S\sigma\right)N = \frac{d\overline{n}}{dz}(z)W - G\overline{n}(z)\left(\frac{d^2}{dz^2} - \kappa^2\right)W\,,$$

where

$$R = \frac{g\widehat{n}_0 v\Delta\rho H^3}{\nu D\rho}\,, \quad S = \frac{\nu}{D}\,, \quad G = \frac{V_sB}{2H}\,.$$

Write down the boundary conditions on W and N for the case in which a no-slip condition applies at both $z = 0$ and $z = -1$.

(c) Seek the critical value R_c of R, above which instability will occur, in the limit $h \to 0$, by assuming a large wavelength and that the eigenvalues σ are real (so that a neutrally stable solution has $\sigma = 0$). Solve the problem by taking

$$\kappa = h\kappa', \quad W = h\sum_{n=0}^{\infty} h^n W_n(z), \quad N = \sum_{n=0}^{\infty} h^n N_n(z), \quad R_c = h^{-1}\sum_{n=0}^{\infty} h^n R_n \,.$$

Show that the leading-order equations give

$$N_0 = \textit{constant} = 1 \text{ (without loss of generality)}$$

$$W_0 = -\frac{\kappa'^2 R_0}{24}(z^4 + 2z^3 + z^2)\,.$$

Then show that

$$N_1 = -GW_0 + z$$

and deduce from the equation for N_2 and the boundary conditions that $R_0 = 720$ for all values of κ'. [*There is no need to calculate* W_1 *explicitly.*]

END OF PAPER