UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2009 9:00 am to 12:00 pm

PAPER 8

ANALYSIS OF OPERATORS

Attempt no more than ***THREE*** *questions.*

There are ***SEVEN*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define what is meant by a Fredholm operator of index n on Hilbert space. Show that an operator is a Fredholm operator of index zero if and only if it is the sum of an invertible operator and a compact operator.

(b) Show that an operator is Fredholm if and only if it is invertible modulo the compact operators.

(c) Show that the Fredholm operators are open in $B(H)$ and that the index is norm continuous. Hence or otherwise deduce that $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$ if S and T are Fredholm operators.

(d) State and prove an index theorem for Toeplitz operators on the Hardy space of a circle.

2 **(a)** State and prove the spectral theorem for compact self–adjoint operators.

(b) Prove that every compact operator on a Hilbert space has a unique polar decomposition.

(c) State and prove von Neumann's double commutant theorem.

(d) Let A be a *–subalgebra of the compact operators on a Hilbert space H and suppose that AH is dense in H. Prove that H breaks up as a direct sum of irreducible closed subspaces for A, each appearing with finite multiplicity.

3 **(a)** Define the Sobolev spaces $H_s(T)$ for $T = \mathbb{T}^n$.

(b) Prove that $H_{s+k}(T)$ lies in $C^k(T)$ for $s > n/2$.

(c) Prove that $H_s(T)$ is invariant under multiplication by smooth functions and that it forms a Banach algebra if $s > n/2$.

(d) If Δ is the Laplacian $-\sum \partial_{x_i}^2$ on T and $V \in C^\infty(T)$, prove that if $u \in H_{-\infty}(T) = \bigcup H_s(T)$ satisfies $(\Delta + V)u = f$ with $f \in C^\infty(T)$, then u must also be smooth.

4 **(a)** State and prove the Krein–Milman theorem for a compact convex subset of a finite–dimensional real vector space. State *without proof* its generalisation to weak* compact convex subsets of the unit ball of the dual of a separable Banach space.

(b) Prove that every self–adjoint operator with non–negative spectrum has a unique square root with the same property.

(c) Let A be a norm–closed separable unital Abelian *–subalgebra of $B(H)$, with H a Hilbert space. Prove that there is a compact space X such that A can be identified with $C(X)$ as a normed *–algebra.

(d) Prove the spectral mapping theorem for polynomials in T and T^*, when T is a normal operator.

5 **(a)** State and prove the Weyl–von Neumann theorem for self–adjoint operators.

(b) Define the essential spectrum of a bounded operator on a Hilbert space. Prove that it is closed and non–empty. (You may assume the axiom of choice if required.)

(c) Let (a_n) and (b_n) be bounded sequences in $\mathbb{R}$ with the same sets of limit points. Prove that there are renumberings (a_{m_k}) and (b_{n_k}) such that $k \in \{m_1, \ldots .m_{2k}\} \cap \{n_1, \ldots, n_{2k}\}$ and $|a_{m_k} - b_{n_k}| \to 0$ as $k \to \infty$.

(d) Prove that two self–adjoint operators are unitary equivalent modulo the compact operators if and only if they have have the same essential spectrum.

6 **(a)** Describe without proof the irreducible finite–dimensional unitary representations of $G = SU(2)$. Explain how the Lie algebra and the Casimir operator of $SU(2)$ acts.

(b) Let H and H' be Hilbert spaces on which G acts unitarily, each decomposing as a direct sum of irreducible representations of finite multiplicity. Let $S : H \to H'$ be a Fredholm operator commuting with G. Define the G–index of S and explain how it can be computed in terms of H and H'.

(c) Define the Dirac operator on S^2 and show that its phase V defines a unitary operator commuting with G and commuting with $C(S^2)$ modulo the compact operators.

(d) If $P(z)$ is the projection on $S^2 = \mathbb{C} \cup \{\infty\}$ defined by

$$P(z) = \begin{pmatrix} \frac{1}{|z|^2+1} & \frac{z}{|z|^2+1} \\ \frac{\overline{z}}{|z|^2+1} & \frac{|z|^2}{|z|^2+1} \end{pmatrix},$$

compute the index of PVP. (You may assume any general results about representations of compact groups that you require, provided they are precisely stated.)

7 **(a)** Define $K^1(X)$ when X is a compact metric space.

(b) Explain how $K^1(X)$ can be realised using continuous maps into $GL(V)$ for V a finite–dimensional complex inner product space.

(c) Show that $K^1(X)$ is an Abelian group.

(d) Prove that $K^1(S^1) \cong \mathbb{Z}$. What can be said about $K^1(S^n)$ for $n > 1$?

END OF PAPER