UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2009 1:30 pm to 4:30 pm

PAPER 79

GEOPHYSICAL AND ENVIRONMENTAL FLUID DYNAMICS

*You may attempt **ALL** questions.*
*Full marks can be achieved by good answers to **THREE** questions.*

Completed answers are preferred to fragments.

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A plane internal gravity wave of frequency ω_i, wavelength λ_i and amplitude η_i is propagating downward through an inviscid, stratified fluid in $z > 0$. The buoyancy frequency for $z > 0$ is N_1, but this changes to N_2 for $z < 0$. The background density $\rho_0(z)$ is continuous at $z = 0$.

(a) By linearising the equations of motion, derive the dispersion relationship for the waves, stating any assumptions made. Give an expression for θ_1 and θ_2, the angle between the vertical and the group velocity of the waves in each of the layers. Describe the geometric relationship between the directions of the phase velocity and group velocity, and any limits on the wave frequency.

(b) Show that if the pressure p is continuous at $z = 0$, then the horizontal velocity component u must also be continuous. Determine the properties of the wave transmitted into $z < 0$, and show that the wave reflected from $z = 0$ has an amplitude η_r satisfying

$$\frac{\eta_r}{\eta_i} = \frac{\frac{\sin\theta_2}{\sin\theta_1} - \frac{\cos\theta_2}{\cos\theta_1}}{\frac{\sin\theta_2}{\sin\theta_1} + \frac{\cos\theta_2}{\cos\theta_1}}.$$

(c) Suppose a rigid boundary is introduced at $z = -H$. Describe how this changes the structure of the disturbance. For $\theta_1 = \pi/3$ and $\theta_2 = \pi/6$, determine how the amplitude of the disturbance varies with H and show that for $-H < z < 0$ the maximum amplitude of the downward propagating wave is $\sqrt{3}\eta_i$. For what H does this occur? What is the amplitude of the upward propagating wave in $z > 0$?

2 Consider a channel in which the cross-sectional area containing water is given by $S(x, h)$, where t is time, x is the along-channel coordinate, $h = h(x, t)$ is the height of the water surface above some datum and $u(x, t)$ is the velocity of the water.

(a) Express the complete derivative dS/dx in terms of partial derivatives and explain the meaning of each term. Derive the continuity equation for this channel, stating any assumptions made. Determine the speed of the characteristics for the flow and show that along the characteristics

$$2\frac{g}{dc^2/dh}\frac{dc}{d\xi} \pm \frac{du}{d\xi} = -\frac{uc}{S}\frac{\partial S}{\partial x},$$

where g is gravity, c is the wave speed, ξ is the time along the characteristics, and $\frac{\partial S}{\partial x}$ is evaluated at constant h.

(b) By linearising around surface height h_0 and area $S_0 = S(x, h_0)$, show that the amplitude $\eta(x)$ of a linear standing wave disturbance in the channel is governed by

$$\frac{d^2\eta}{dx^2} + \frac{S_0'}{S_0}\frac{d\eta}{dx} + k^2\eta = 0,$$

where $S_0' = dS_0/dx$ and k is the wavenumber.

(c) For the case of a channel with $S_0' = \beta S_0$ with constant β, discuss how the form of the solution depends on the magnitude of k. Determine the wavenumber of the standing modes that can exist in a basin of length L bounded by dams at each end.

(d) Describe the flow that would develop if the dam at one end of the channel were to fail at $t = 0$. Assuming the channel close to the dam that fails has a triangular cross-section of uniform depth and width, determine the depth profile and front speed of the current that develops.

3 A volume V_0 of water of density ρ_0 containing bubbles with volume concentration $\phi_0 \ll 1$ is released at $z = 0$ beneath the surface of the ocean to form a bubbly thermal. The density of the ocean is $\rho_a(z)$ with $\rho_a(0) = \rho_0$, and the bulk density of the thermal is $\rho(1-\phi)$ where ρ is the density of the water in the thermal and ϕ is the local volume concentration of bubbles. As the thermal rises, Boyle's law states that for isothermal bubbles the density of the gas in each bubble obeys $p/\rho_b = p_0/\rho_B$, where p is the pressure and ρ_b is the gas density. Here, p_0 and ρ_B are the pressure and gas density at $z = 0$.

(a) Determine the variation with depth of the gas density and hence the contribution of the bubbles to the buoyancy of the thermal. In this calculation you may neglect variations in $\rho_a(z)$ and the influence of surface tension, and assume the gas density is small. Under what conditions can the rise velocity of the bubbles be neglected?

(b) State the Boussinesq assumption and describe the entrainment coefficient α and the mechanism by which the volume of the thermal increases as it rises. Assuming ϕ remains small and the thermal is spherical, show that the diameter $2a$ of the thermal grows linearly with height, independently of the stratification.

(c) Justify the use of a Froude number condition based on the diameter of the thermal for the rise velocity of the thermal. What is the principal drag mechanism? Show that for $z \gg a_0/\alpha$ in a homogeneous ocean the rise velocity is

$$W = F\left(\frac{2\phi_0 a_0^3 g}{\left(1 - \frac{\rho_0 g}{p_0} z\right)\alpha^2 z^2}\right)^{\frac{1}{2}},$$

where F is a Froude number and $2a_0$ is the initial diameter of the thermal.

(d) For a stratified ocean described by constant buoyancy frequency N, determine the density ρ of the water within the thermal as a function of z and hence the bulk reduced gravity for the thermal. Show that the maximum rise height of the thermal is given by the solution of a fifth order polynomial. [You need not determine the solution of this polynomial.]

(e) Give an explicit solution for the rise height in the case of a very deep release. Describe briefly what you would expect to happen after the thermal reaches this height?

4 A tall tube of height $2H$ and square cross-section $W \times W$ ($W \ll H$) is filled with a statically unstable stratification given by $\rho_0(1 + G(z))$ at $t = 0$, where ρ_0 is the density at $z = 0$, and $G(z)$ is an odd function that increases monotonically with z. The flow that develops may be characterised by high Reynolds number turbulent diffusion driven by the release of potential energy from the density stratification.

(a) Sketch the form of the energy density spectrum for high-Reynolds-number, homogeneous, isotropic turbulence and indicate key features. Discuss how this relates to the flow that might develop in the tube. Show how Reynolds stresses $\overline{u_i u_j}$ arise from the Navier-Stokes equations and discuss how these may be modelled through a turbulent viscosity.

(b) Using dimensional analysis, determine a typical velocity scale for the turbulence in the tube through an instantaneous balance between buoyancy and inertial forces. Define a suitable turbulent mass diffusivity and show that the flux of density across any horizontal plane in the tube is given by

$$F_\rho = K \left(\frac{g}{\rho}\right)^{\frac{1}{2}} W^2 \left(\overline{\frac{\partial \rho}{\partial z}}\right)^{\frac{3}{2}},$$

where K is a constant and $\overline{\frac{\partial \rho}{\partial z}}$ is the density gradient averaged over the horizontal plane. State any assumptions made. Write down the equation for the evolution of the vertical density gradient (averaged over a horizontal plane).

(c) Show that the density gradient in a Boussinesq fluid will increase if

$$\left(\frac{\partial^2 \bar{\rho}}{\partial z^2}\right)^2 + 2\frac{\partial \bar{\rho}}{\partial z}\frac{\partial^3 \bar{\rho}}{\partial z^3} > 0.$$

Suppose the horizontally-averaged density profile is self-similar and given by $\bar{\rho}(z,t) = \rho_0(1 + F(t)G(z))$. Determine a nonlinear form for $G(z)$ that gives self-similarity in a Boussinesq fluid and hence determine the evolution of the density field within the tube. [You need not take into account the boundary conditions at $z = \pm H/2$.] Comment on the validity of the model as t increases.

5 A single layer of fluid flows along a channel of width $b(x)$ and bottom elevation $H(x)$. The channel is rotating with angular frequency $f/2$ about the vertical z axis.

(a) Write down an expression for potential vorticity P and show that this is related to the Bernoulli potential B for an inviscid shallow water flow by

$$dB/d\psi = -P,$$

where the horizontal velocity is given by $\mathbf{u}h = \nabla \wedge (\psi \hat{\mathbf{z}})$, h is the layer depth, ψ is a stream function and $\hat{\mathbf{z}}$ is the unit vector in the vertical direction. How does P evolve for a fluid element in an inviscid flow?

(b) Assume the flow is in geostrophic balance with negligible cross-channel velocity and show that the volume flux along the channel is given by

$$Q = \frac{g}{2f}\left(h_1^2 - h_2^2\right),$$

where h_1 and h_2 are the surface heights at the two walls, $y = 0$ and $y = b$, respectively.

(c) Taking the velocity at $y = 0$ as $u_1(x)$, determine how the depth and velocity of the flow vary across the channel for a steady flow from a reservoir in which $P = 0$. Describe the conditions in such a reservoir.

(d) Show that the velocity u_1 at $y = 0$ is given by

$$u_1 = -\frac{bf}{2} + \frac{gh_1}{bf} - \left(\frac{g^2h_1^2}{b^2f^2} - 2\frac{Qg}{b^2f}\right)^{\frac{1}{2}},$$

and hence derive a specific energy function E for this wall. How does the specific energy vary along the channel? How does it vary across the width of the channel?

(e) Suppose that a hydraulic control exists at $x = 0$. What is the significance of the hydraulic control? Determine Q in terms of h_1 at this point. What limit has to be placed on b for this analysis to be valid?

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