UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2009 9:00 am to 12:00 pm

PAPER 78

WAVES IN FLUIDS

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the rigid body whose surface is defined by $F(\boldsymbol{x},t)=0$, with $F>0$ outside the body. Starting from the kinematic condition

$$\frac{\partial F}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla} F = 0 \ ,$$

and the equations of mass and momentum conservation for an inviscid fluid, show that

$$\left[\frac{\partial^2}{\partial t^2} - c_0^2\boldsymbol{\nabla}^2\right]\left[\rho'\mathrm{H}(F)\right] = \frac{\partial}{\partial t}\left\{\rho_0 \boldsymbol{u}.\boldsymbol{\nabla} F\delta(F)\right\} - \frac{\partial}{\partial x_i}\left\{p\frac{\partial F}{\partial x_i}\delta(F)\right\} + \frac{\partial^2}{\partial x_i \partial x_j}\left\{T_{ij}\mathrm{H}(F)\right\} \ ,$$

where ρ' is the acoustic density perturbation to the mean density ρ_0, p is the fluid pressure, T_{ij} is the Lighthill quadrupole (to be defined) and $\mathrm{H}(z)$ is the unit step function. Show how ρ' can be written as the sum of three integrals. In the case in which the acoustic sources are compact, show how these integrals can be simplified to yield a result of the form

$$\rho'(\boldsymbol{x},t) = \frac{\dot{m}(t-|\boldsymbol{x}|/c_0)}{4\pi|\boldsymbol{x}|c_0^2} - \frac{\hat{x}_i\dot{f}_i(t-|\boldsymbol{x}|/c_0)}{4\pi|\boldsymbol{x}|c_0^3} + \frac{\hat{x}_i\hat{x}_j\ddot{S}_{ij}(t-|\boldsymbol{x}|/c_0)}{4\pi|\boldsymbol{x}|c_0^4} \ , \qquad (1)$$

where the quantities m, f_i and S_{ij} are to be defined. Be careful to explain clearly the approximations which have been used to derive equation (1).

Consider a small compact sphere of constant radius a, whose centre undergoes small-amplitude oscillations of amplitude a_1 and frequency ω. Show how the various components of the far-field sound scale with the small parameter $a\omega/c_0$. Calculate explicit expressions for the components of the sound associated with the terms $\dot{m}$ and $\dot{f}_i$ in equation (1).

[*The free-space Green's function is*

$$G(\boldsymbol{x},t) = \frac{\delta(t-|\boldsymbol{x}|/c_0)}{4\pi|\boldsymbol{x}|c_0^2} \ . \qquad]$$

2 Consider the semi-infinite waveguide formed by the rigid plates $x > 0, y = \pm b$. The incident plane wave with velocity potential

$$\phi_{inc} = \exp(\mathrm{i}k_0 x + \mathrm{i}\omega t)$$

propagates in the negative x direction inside the waveguide. Show that the Fourier transform of the scattered potential can be written in the form

$$\frac{\mathrm{i}L^+(-k_0)L^-(k)\cosh(\gamma y)}{\gamma \sinh(\gamma b)(k + k_0)} \quad \text{for } |y| < b \,,$$

where k is the Fourier-transform variable, $\gamma(k) = \sqrt{k^2 - k_0^2}$ and $L(k) = \gamma \sinh(\gamma b) \exp(-\gamma b)$. The factors $L^{\pm}(k)$ satisfy $L(k) = L^+(k)L^-(k)$ and are analytic, nonzero and have algebraic behaviour at infinity in the upper and lower halves of the complex plane respectively.

Find a corresponding expression for the Fourier transform of the scattered potential in $|y| > b$.

Determine the following:

(i) the far-field scattered potential in $|y| > b$;

(ii) the amplitude of the plane wave reflected back down the wave guide in the positive x direction.

Show that the total potential (incident plus scattered) is proportional to $(-x)^{1/2}$ as $x \to -\infty$ in $|y| < b$.

[*Hint: Use the Fourier transform convention*

$$\Phi(k, y) = \int_{-\infty}^{\infty} \phi(x, y) \exp(\mathrm{i}kx)\mathrm{d}x \,.$$

You may use the result

$$\int_\Gamma f(k) \exp(-\mathrm{i}kr \cos\theta - \gamma r |\sin\theta|)\mathrm{d}k \sim \left(\frac{2k_0\pi}{r}\right)^{1/2} f(k_0 \cos\theta) |\sin\theta| \exp(-\mathrm{i}k_0 r + \mathrm{i}\pi/4)$$

as $r \to \infty$, where Γ is the steepest descent contour.]

3 (a) The evolution of a plane wave is governed by Burgers equation

$$\frac{\partial q}{\partial X} - q\frac{\partial q}{\partial \theta} = \epsilon\frac{\partial^2 q}{\partial \theta^2} ,$$

with $q(\theta, 0) = f(\theta)$ given. When $\epsilon = 0$ obtain the characteristic solution, and briefly explain how to use weak shock theory when the wave overturns. What happens when

$$f(\theta) = \begin{cases} 0 & \theta < 0 \\ U & \theta > 0 , \end{cases} \tag{1}$$

for $U > 0$?

For $\epsilon > 0$ use the Cole-Hopf transformation

$$q = 2\epsilon\frac{\partial}{\partial \theta}\ln\psi ,$$

together with the general solution of the diffusion equation,

$$\psi(\theta, X) = \frac{1}{(4\pi\epsilon X)^{1/2}}\int_{-\infty}^{\infty}\psi(\theta', 0)\exp\left[-\frac{(\theta-\theta')^2}{4\epsilon X}\right]\mathrm{d}\theta' ,$$

to find the solution of Burgers equation with initial data given by equation (1). You may write your answer in terms of the complementary error function

$$\mathrm{erfc}(z) \equiv \frac{2}{\sqrt{\pi}}\int_z^{\infty}\exp(-t^2)\mathrm{d}t .$$

(b) Let $\mathbf{E}$ be a time-harmonic electromagnetic wave propagating in a 3-dimensional medium with permittivity $\epsilon = \epsilon(x)$ and permeability $\mu = \mu_0$, where μ_0 is the permeability of free space, so that the wave equation for E can be written as

$$\nabla^2\mathbf{E} + k^2\epsilon_{rel}\mathbf{E} = 0 , \tag{2}$$

where $\epsilon_{rel} = \epsilon(x)/\epsilon_0$, with ϵ_0 the permittivity of free space.

(i) Derive the parabolic wave equation for

$$U(x, y, z) = E(x, y, z)e^{-ikx} , \tag{3}$$

where E is a scalar field propagating at a small angle with respect to the x-direction. Explain under what condition it is a good approximation and what physical effects may be neglected.

(ii) Consider now a linearly polarized electromagnetic wave, such that the electric field vector lies in the (x, z) plane: $\mathbf{E} = E(x, y, z)\hat{\mathbf{e}}$, where $\hat{\mathbf{e}} = (1, 0, 1)$. Assume that this wave is propagating at a small angle with respect to the x-direction, and we can write

$$\mathbf{E} = \mathbf{U}e^{ikx} = U(x, y, z)\hat{\mathbf{e}}e^{ikx} \tag{4}$$

and the slowly-varying function $U(x, y, z)$ obeys the same parabolic equation as derived in (i). Show that the total energy flux at any plane $x = const$ must be constant.

[*Hint: You may wish to define the Hermitian operator*

$$L = \frac{1}{2k}\nabla_\perp^2 + \frac{k}{2}(\epsilon_{rel} - 1) \ , \quad where \ \nabla_\perp^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{5}$$

in Green's theorem:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(Lg)^* ds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^*(Lf) ds \ , \tag{6}$$

with f and g sufficiently smooth functions, chosen here as $f = g = E(x, y, z)$, and to use the relationship $E = \sqrt{\mu/\epsilon}H$ between the amplitudes of the electric and magnetic field of a plane wave.]

4 Consider a 2-dimensional space with horizontal coordinate x and vertical coordinate z. Let $z = h(x)$ define a perfectly reflecting, statistically rough surface with mean $< h >= 0$ and r.m.s. height σ, with the medium above the surface having density ρ.

A time-harmonic monochromatic acoustic plane wave ψ_i with frequency ω and corresponding wavenumber k is incident upon the surface at an angle θ with the horizontal.

(a) In the case of small surface height, $|kh(x)| \ll 1$, derive an expression for the scattered field $\psi_s(x, z)$. Assuming that the surface $z = h(x)$ is statistically stationary, derive the mean scattered field.

(b) Write a general expression for the mean scattered field $< \psi_s(x, z) >$ in terms of an effective (unknown) reflection coefficient. Now consider an additional, flat, impedance surface at $z = d$, defining a layer $d \geqslant z \geqslant h(x)$, so that the medium above the rough surface $h(x)$ is divided into the layer, with density ρ, and an upper medium $z > d$ with density ρ_2 and corresponding wavenumber k_2.

Write an expression for the mean field $< \psi_{tot}(x, z) >$ in the layer.

5 Given the direct scattering problem defined by

$$Ax = y \,, \tag{1}$$

where A is a symmetric positive definite $n \times n$ matrix, A has the spectral representation

$$Ax = \sum_{i=1}^{n} \lambda_i (x, u_i)\, u_i \,, \tag{2}$$

where λ_i are eigenvalues $0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_n$ and u_i are the corresponding orthonormal eigenvectors.

Consider now the inverse problem

$$x = A^{-1} y \,. \tag{3}$$

Assume that $\lambda_1 \neq 0$ and that the known data y_δ is given with a known error δ such that

$$\| \, y_\delta - y \, \| \leqslant \delta \,. \tag{4}$$

(a) Relate the stability of the inverse problem (**??**) to the value of the ratio between the largest and the smallest eigenvalues $\kappa = \lambda_n / \lambda_1$.

(b) Describe how Tikhonov regularisation is applied to the inverse problem (**??**) to find a regularised solution $x_\alpha = R_\alpha y$.

Estimate the error between the exact solution of the inverse problem $x = A^{-1}y$ and the regularised solution $x_{\alpha(\delta)}$ of the regularised inverse problem with real data $x_{\alpha(\delta)} = R_\alpha y_\delta$, showing its dependence on the error in the data, δ .

(c) Now assume that you use a different, simpler regularisation, defined by the operator $S_\alpha = (\alpha I + A)^{-1}$, so the regularised solution from real data is now $x_{\alpha(\delta)} = S_\alpha y_\delta$. Use the spectral representation of the operator A , and of S_α to derive an expression for the error between the exact and the regularised solution which explicitly relates α and δ . Choose a reasonable upper bound for $\| \, x_{\alpha(\delta)} - x \, \|$, hence choose α .

END OF PAPER