UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2009 9.00 am to 12.00 pm

PAPER 76

PERTURBATION AND STABILITY METHODS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) In the limit $\epsilon \to 0$, with ϵ real, find two terms of an asymptotic expansion for each of the roots, both real and complex, of the equation

$$\frac{z}{(1+z^2)^2} = \epsilon\,.$$

(b) Fresnel functions $C(\lambda)$ and $S(\lambda)$ are defined for real $\lambda > 0$ as

$$C(\lambda) = \int_0^{\lambda} \cos(t^2)dt \quad S(\lambda) = \int_0^{\lambda} \sin(t^2)dt\,.$$

Obtain full asymptotic expansions for $C(\lambda)$ and $S(\lambda)$ in each of the limits $\lambda \to 0$ and $\lambda \to \infty$, giving the first three terms of the expansions for $C(\lambda)$ explicitly. [*Standard results may be quoted without proof.*]

A pocket calculator has the usual 'built-in' functions. Based on your results above, how would you construct an approximation to $C(\lambda)$ valid for *all* $\lambda > 0$ suitable for such a calculator? [*Detailed calculation to obtain such a formula is NOT required nor any arithmetic calculation of* $C(\lambda)$.]

2 Use the method of multiple scales to solve the following problems.

(i) A perturbed oscillator satisfies the equation

$$\frac{d^2x}{dt^2} + x = \epsilon f\left(x, \frac{dx}{dt}\right) \quad \text{with} \quad \epsilon \ll 1\,.$$

If the solution of the equation is written

$$x = R(T)\cos(t + \phi(T)) \quad \text{with} \quad T = \epsilon t$$

show that

$$\frac{dR}{dT} = -\langle f \sin(t+\phi)\rangle \quad R\frac{d\phi}{dT} = -\langle f\cos(t+\phi)\rangle$$

where the average $\langle \ldots \rangle$ should be defined.

Find R and ϕ explicitly if $f = -x^3$ with arbitrary initial conditions for x.

[*Hint:* $\cos^4\theta \equiv \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$.]

(ii) Derive similar averaged results for the coupled problem

$$\frac{d^2x}{dt^2} + x = -2\epsilon y\frac{dx}{dt} \qquad \frac{dy}{dt} = \epsilon g(x)\,.$$

Hence find $R(T)$ and $\phi(T)$ explicitly for the cases $g(x) = x$ and $g(x) = \frac{1}{2}\log(x^2)$, with arbitrary initial conditions for x and y.

$$\left[\int_0^{2\pi} \log\left(\cos^2\theta\right) d\theta = -4\pi\log 2\right]\,.$$

3 Rayleigh's equation,

$$\left(\frac{d^2}{dy^2} - k^2\right)\phi - \left(\frac{U''}{U-c}\right)\phi = 0 \ ,$$

governs the growth of linear disturbances on a two-dimensional flow

$$\mathbf{u} = (U(y), 0) \, ,$$

where ϕ is the disturbance eigenfunction, k is the wavenumber, c is the wavespeed and $U'' = \frac{d^2U}{dy^2}$. Assume that there are rigid walls at $y = a < 0$ and $y = b > 0$, so that

$$\phi = 0 \quad \text{on} \quad y = a \text{ and } y = b.$$

Suppose for a given smooth monotonic profile $U(y)$ (with $U'(y) > 0$), that a *neutral mode* can be found with real regular eigenfunction $\phi = \phi_0(y)$, real wavenumber $k = k_0$ and real wavespeed $c = c_0$. You may assume both that there exists a unique value $y = y_0$ such that $U(y_0) = c_0$, and that $\phi_0(y_0) \neq 0$. Give a very brief argument why, in general, $U''(y_0) = 0$.

Henceforth assume that a and b have been chosen so that $y_0 = 0$, and that $U''(0) = 0$.

Consider asymptotic solutions to Rayleigh's equation when $|k - k_0| \ll 1$ by writing

$$k = k_0 + \epsilon k_1 + \dots \, , \qquad c = c_0 + \epsilon c_1 + \dots \, , \quad \phi = \phi_0 + \epsilon \phi_1 + \dots \, ,$$

where $0 < \epsilon \ll 1$. Deduce that

$$\left(\frac{d^2}{dy^2} - k_0^2\right)\phi_1 + \frac{U''}{U-c_0}\phi_1 = \left(2k_1k_0 + \frac{c_1U''}{(U-c_0)^2}\right)\phi_0 \, . \qquad (*)$$

Assume that $\mathrm{Im}(c_1) > 0$, and that as $y \to 0$:

$$\phi_0 = \phi_0(0) + y\phi_0'(0) + \dots \qquad U = c_0 + yU'(0) + \tfrac{1}{6}y^3U'''(0) \dots \, .$$

By using equation $(*)$ deduce that as $y \to \pm 0$:

$$\phi_1(y) = \phi_1(0) + y \log|y| \beta_\pm + y b_\pm + \dots$$

and obtain expressions for β_+ and β_- in terms of c_1, $U'(0)$ and $U'''(0)$.

Explain why it is appropriate to introduce an 'inner' scaling $y = \epsilon^\ell \eta$ near $y = 0$, where ℓ is to be identified. In terms of the variable η assume in the 'inner' region that

$$\phi = \Phi_0(\eta) + \epsilon\Phi_1(\eta) + (\epsilon^2 \log \epsilon)\, \theta_2(\eta) + \epsilon^2\Phi_2(\eta) + \dots \ .$$

By finding expressions for Φ_0, Φ_1, etc., show that

$$b_+ - b_- = \frac{i\pi c_1 U'''(0)\phi_0(0)}{U'(0)^2} \, .$$

Hint. You may assume that a particular solution to

$$\Phi''(\eta) = A + \frac{B\eta}{D\eta - C}$$

is

$$\Phi = \tfrac{1}{2}\left(A + \frac{B}{D}\right)\eta^2 + \frac{BC}{D^3}\left((D\eta - C)\log(D\eta - C) - (D\eta - C)\right) .$$

By using equation $(*)$ show that for $\delta > 0$

$$\left(\int_a^{-\delta} + \int_\delta^b\right)\left(2k_1k_0 + \frac{c_1U''}{(U-c_0)^2}\right)\phi_0^2 dy = \Big[\phi_1\phi_0' - \phi_0\phi_1'\Big]_{-\delta}^{\delta} .$$

Hence, by considering the limit $\delta \to 0$, deduce that k_1 and c_1 are related by

$$2k_1k_0\int_a^b \phi_0^2 dy = -c_1\left(\int_a^b \frac{U''}{(U-c_0)^2}\phi_0^2 dy + \frac{i\pi U'''(0)\phi_0(0)^2}{U'(0)^2}\right) ,$$

where you should assume that the integrals are defined.

4 Consider an inviscid fluid in an unbounded two-dimensional domain. Suppose that for times $t < 0$ the fluid motion is described by Couette flow with the velocity profile

$$\mathbf{u} = (y, 0) .$$

Show that the equation governing linearised disturbances to this flow is

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = 0 , \qquad (\dagger)$$

where $\psi(x, y, t)$ is the streamfunction of the disturbance.

Suppose that at $t = 0$ there is an initial disturbance $\psi = \exp(i\alpha x)\phi(y)$, where $\alpha > 0$, ϕ is a smooth function, and $\phi \to 0$ as $|y| \to \infty$. Show for $t > 0$ that the solution for ψ can be expressed as

$$\psi = \int_{-\infty}^{\infty} F(y_0)G(y, y_0)\exp(i\alpha(x - y_0t))dy_0 ,$$

where you may assume that the integral converges and you should identify the functions $F(y_0)$ and $G(y, y_0)$.

Show by integration by parts, or otherwise, that in general $\psi(x, y, t) = O(t^{-2})$ as $t \to \infty$. Also deduce the large time behaviour of (a) $u = \psi_y$, (b) $v = -\psi_x$ and (c) $\omega = -(\psi_{xx} + \psi_{yy})$.

For an initial disturbance $\omega(x, y, 0) = \omega_0(x, y)$ solve for ω directly from equation $(\dagger)$. Confirm the large time behaviour of ω deduced in (c) above.

END OF PAPER