UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2009 1:30 pm to 4:30 pm

PAPER 74

SLOW VISCOUS FLOW

*Attempt **QUESTION 1** and at most **TWO OTHER** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 *You do not need to calculate the details of any specific flow to answer this question.*

(i) State carefully the minimum dissipation theorem for Stokes flow.

(ii) A dense sphere of radius a is falling down a very long, vertical tube of radius $R > 2a$ filled with viscous fluid, and is the only cause of flow in the tube. Explain carefully why the net volume flux of fluid must be upwards in any cross-section containing the particle.

(iii) Criticise the statement that, if it is not already there, the sphere will migrate to the centreline of the tube as it falls because this position minimizes the dissipation. Is the statement about migration true even though the reason given is false?

(iv) The dense sphere is on the centreline and a second neutrally buoyant sphere of radius a is added somewhere to the flow. Show carefully that the vertical component of velocity of the dense sphere in the initial centreline position is less than it was with the second sphere absent.

[*Hints: You may assume that the resistance matrix, giving the force and couple on a single sphere in the tube in terms of its velocity and angular velocity, is diagonal with respect to axes aligned with the tube when the sphere is on the centreline. You should not assume that the dense sphere moves vertically without rotation when the second sphere is present.*]

(v) Initially, the first sphere is a long way above the second sphere and is on the centreline of the tube. The centre of the second sphere is initially a distance $\frac{1}{10}a$ from the centreline. Use a scaling argument based on lubrication theory to explain why the two spheres do not in fact touch as the first sphere falls past the second.

(vi) Show further that the spheres return to their original distances from the centreline when the first sphere has fallen to a long way below the second.

(vii) The second sphere is now made as dense as the first, and $a \ll R$. Use part (ii) to explain why the first sphere may now fall more rapidly or more slowly than the original rate in the same position, depending on the relative configuration.

2 (a) State the Papkovich–Neuber representation for the velocity and pressure in Stokes flow.

Use the Papkovich–Neuber representation, explaining your choice of trial harmonic potential, to determine the velocity and stress fields due to a point force $\mathbf{F}\delta(\mathbf{x})$ at the origin of unbounded fluid of viscosity μ.

(b) An inviscid spherical bubble of radius a and negligible density rises with velocity $\mathbf{U}$ through unbounded fluid of density ρ and viscosity μ that is otherwise at rest.

State the boundary conditions on the bubble. Show that these conditions can be satisfied by an appropriate choice of $\mathbf{F}$ in the Stokeslet flow of part (a), now understood to apply only in the region $r > a$. Deduce that the speed of rise is given by

$$U = \frac{\rho g a^2}{3\mu}.$$

Does the bubble remain spherical if there is no surface tension? Justify your answer briefly.

(c) Suppose the bubble is now rising through a semi-infinite body of fluid that has a free surface at a distance d above the centre of the bubble, where $d \gg a$. Owing to the motion of the bubble, the free surface is deflected upwards from its equilibrium position until the restoring effect of gravity is just sufficient to ensure that the normal velocity $\mathbf{u}\cdot\mathbf{n}$ is approximately zero on the surface. Use scaling arguments to show that the deflection h is $O(a^3/d^2)$.

Hence explain why the flow in this problem is approximately the same as that in an unbounded fluid when there is an inviscid drop of density 2ρ and radius a at a distance $2d$ above the bubble. Deduce that the velocity of the bubble is approximately $U(1 - \frac{1}{2}a/d)$.

What dimensionless group needs to be small if surface tension is to keep the bubble roughly spherical?

3 Crude oil exudes from a long slit in the side of a wrecked tanker to form a two-dimensional gravity current on the sea-surface. The viscosity of the oil μ is very much greater than that of the sea-water, and its density ρ is less than the density ρ_w of the water. The density of air can be neglected. The horizontal extent of the current is much greater than its thickness.

Show that the gravity current is driven by a pressure gradient

$$\frac{\partial p}{\partial x} = \rho g' \frac{\partial h}{\partial x}, \qquad \text{where} \quad g' = \frac{\rho_w - \rho}{\rho_w} g, \tag{$*$}$$

$h(x,t)$ is the thickness of the current and x is the horizontal distance from the side of the tanker.

Derive from first principles the equations

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0, \qquad 4\mu\frac{1}{h}\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right) = \frac{\partial p}{\partial x},$$

where $u = u(x,t)$ is the horizontal velocity in the current.

[*The effect of gravity in the horizontal momentum balance may* either *be assumed simply to be a horizontal body force equal to minus the driving pressure gradient* $(*)$ or *obtained by integrating the hydrostatic pressure over the sides and bottom of a slice.*]

The flow through the slit provides a constant thickness h_0 and velocity u_0 at $x = 0$. You may assume that the boundary condition at the nose of the current is

$$4\mu h_N \left.\frac{\partial u}{\partial x}\right|_{x_N} = \frac{1}{2}\rho g' h_N^2,$$

where $h_N(t)$ is the thickness and $x_N(t)$ the position at the nose.

Use a Lagrangian approach to determine the thickness, velocity and position at time t of the slice of fluid that left the tanker at time t_0. What is the location $x_N(t)$ of the nose?

Show that the thickness of the current at a given position x is constant after the nose has passed, and determine the shape $h(x)$ of the current. What is the physical significance of the lengthscale $\mu u_0/(\rho g' h_0)$?

4 A snail can crawl forward over a thin layer of mucus by propagating waves backwards along the underside of its foot. Model this by the following idealized problem in lubrication theory (treating mucus as a viscous fluid with viscosity μ).

Let the snail be infinitely long and move horizontally with uniform speed U, to be calculated, over a stationary rigid plane. The waves in the shape of the foot move backwards with prescribed speed V relative to the snail (and thus $V-U$ relative to the plane).

In the frame of reference moving with the waves, the layer of mucus has constant thickness $h(x)$, where h is a prescribed periodic function with wavelength $L \gg h$. Working in this frame of reference, use lubrication theory to determine the fluid velocity in terms of the boundary velocities and the local pressure gradient.

Given that $p(0) = p(L)$, show that the volume flux q in this frame is given by

$$q = \frac{(V - \frac{1}{2}U)I_2}{I_3}, \quad \text{where } I_j = \int_0^L h^{-j}\, dx.$$

Hence determine the local pressure gradient and the shear stress on the plane.

Why is the integrated horizontal force on the plane zero? Show that

$$U = \frac{6(1-\alpha)V}{4-3\alpha}, \quad \text{where } \alpha = \frac{I_2^2}{I_1 I_3}.$$

Find U for the case $h(x) = h_0[1 - A\sin(2\pi x/L)]$, where $0 \leqslant A < 1$. Comment on the form of the result for U in the limits $A \to 0$ and $A \to 1$.

$$\left[\int_0^{2\pi} \frac{d\theta}{(1-A\sin\theta)} = \frac{2\pi}{(1-A^2)^{1/2}} \qquad \int_0^{2\pi} \frac{d\theta}{(1-A\sin\theta)^2} = \frac{2\pi}{(1-A^2)^{3/2}} \right.$$

$$\left. \int_0^{2\pi} \frac{d\theta}{(1-A\sin\theta)^3} = \frac{2\pi(1+A^2/2)}{(1-A^2)^{5/2}} \right]$$

For a general waveform $h(x)$, show that the viscous dissipation Φ per wavelength is equal to $2\mu UVI_1$. [*You may assume that the pressure does no work at the boundaries because the boundary motion is purely tangential in this frame.*]

Consider a snail wishing to crawl at a given speed U using a waveform $h = h_0[1 - A\sin(2\pi x/L)]$. What value of A should an efficient snail use in order to minimize the viscous dissipation Φ? Show that the optimal value of A still gives $8\sqrt{2}/3$ times the viscous dissipation produced by a snail gliding magically with speed U over a uniform film of thickness h_0.

END OF PAPER