UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2009 9:00 am to 12:00 pm

PAPER 73

FLUID DYNAMICS OF ENERGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

This is an ***OPEN BOOK*** *examination.*
Candidates may bring handwritten notes and lecture handouts into the examination.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Buoyant fluid of density $\rho - \Delta\rho$ is injected at rate $Q \exp(-t/\tau)$ per unit length from a horizontal well into an inclined aquifer, of thickness H, permeability k and porosity ϕ filled with fluid of density ρ. The aquifer is bounded above and below by an impermeable layer of rock. The fluid spreads upslope forming a thin plume, of thickness $h(x,t) \ll H$, where x is the alongslope distance. The dynamic viscosity of the fluid is μ.

(i) Show that the current depth evolves according to the relation

$$\phi \frac{\partial h}{\partial t} = \frac{-kg\Delta\rho}{\mu} \sin\theta \left[\frac{\partial h}{\partial x} - \cot\theta \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) \right].$$

(ii) Show that the solution for the shape of the current at sufficiently large time is

$$h(x,t) = h(0,0) \exp\left[\frac{-ut + x\phi}{u\tau} \right], \tag{1}$$

where u is a velocity which should be determined explicitly.

(iii) Determine:

1. an expression which defines the "sufficiently large time" at which this approximation holds;
2. the boundary condition at the nose of the current in this limit.

(iv) If the trailing edge of the current leaves behind a fraction s of the fluid owing to capillary retention, show that the solution (**??**) changes to

$$h(x,t) = h(0,0) \exp\left[\frac{-ut + x\phi(1-s)}{u\tau} \right].$$

(v) Use this solution to calculate an expression for the volume of fluid in the current as a function of time.

(vi) Show that this reaches a maximum value after a finite time, which should be determined.

2 Polymer-laden water is injected into a horizontal well in an oilfield to displace the oil which is extracted from a parallel horizontal well a distance L from the injection well. The polymer-laden water has dynamic viscosity μ. If the flow in the oilfield is stable, it may be described by a flow field $u(x,t)$, (in the direction x perpendicular to the wells) which satisfies Darcy's law. In the injection process, the pressure at the injection well is maintained an amount ΔP greater than that at the production well.

(i) Assume that the interface between the oil and the water is planar, that the dynamic viscosity of the oil is μ_h, that the permeability is k and that the porosity is ϕ. Derive an expression for the speed of the current as a function of time assuming that the concentration of the polymer is constant.

(ii) It is discovered that the influence of the polymer is also sensitive to the temperature of the fluid, so that the dynamic viscosity increases by a factor b as the polymer solution heats up from the injection temperature to the temperature of the field. Under these circumstances, repeat the calculation to determine the rate of advance of the polymer front with time, following the onset of injection. Assume that:

- the advection speed of isotherms is a fraction Γ of the Darcy speed;
- thermal diffusion is very slow;
- the flow is stable across the thermal front.

(iii) Explain why, in practice, the flow may become unstable across the thermal front.

(iv) In the case that the flow speed is constant, derive an expression for the growth rate of a sinusoidal disturbance to the interface as a function of the wavelength of the disturbance, neglecting the effects of thermal diffusion.

3 The $k-\epsilon$ equations in high Reynolds-number, fully developed channel flow (under the assumption that all the quantities of interest depend only on the wall-normal coordinate y) may be written as

$$
\begin{aligned}
0 &= \frac{d}{dy}\left(\frac{\nu_T}{\sigma_k}\frac{dk}{dy}\right) + \mathcal{P} - \epsilon, \\
0 &= \frac{d}{dy}\left(\frac{\nu_T}{\sigma_\epsilon}\frac{d\epsilon}{dy}\right) + C_{\epsilon 1}\frac{\mathcal{P}\epsilon}{k} - C_{\epsilon 2}\frac{\epsilon^2}{k},
\end{aligned}
$$

where the turbulent viscosity $\nu_T = C_\mu k^2/\epsilon$, and σ_k, σ_ϵ, C_μ, $C_{\epsilon 1}$ and $C_{\epsilon 2}$ are empirical constants.

(i) Explain briefly the physical significance of each term in the k-ϵ equations.

(ii) Write down expressions for the turbulence production $\mathcal{P}$ and the turbulent viscosity ν_T in terms of the mean shear and the Reynolds' stress.

(iii) Now consider the flow in the log-law region. Write down the expression for the mean shear in the log-law region.

(iv) Present an argument for why the turbulence production and the dissipation rate should balance in the log-law region such that $\mathcal{P} \simeq \epsilon = u_\star^3/\kappa y$, where κ is the von-Karman constant, and $u_\star$ is the friction velocity, a quantity which you should define carefully.

(v) Therefore, show that the turbulent kinetic energy is independent of y.

(vi) By using the balance of turbulence production and dissipation rate and the equivalent definitions of ν_T in terms of k and ϵ and in terms of the mean shear and Reynolds' stress in the log-law region, show that the dissipation rate may be expressed as

$$
\epsilon = \frac{C_\mu k^2}{u_\star \kappa y} = \frac{C_\mu^{1/2} k u_\star}{\kappa y} = \frac{C_\mu^{3/4} k^{3/2}}{\kappa y}.
$$

(You may find it useful to express the Reynolds' stress in terms of the friction velocity.)

(vii) Therefore, use the ϵ equation to express the von-Karman constant κ in terms of σ_ϵ, C_μ, $C_{\epsilon 1}$, and $C_{\epsilon 2}$.

4 Consider a fully turbulent, high-Reynolds number flow which is statistically steady and horizontally homogeneous, and has a mean velocity in the x-direction and a statically stable mean density gradient which depend on z and possibly time. Assume that the mean density decreases with z, while the mean velocity increases with z. Assuming that diffusion terms are small, and that the Boussinesq approximation is valid, the Navier-Stokes equations and the density equation reduce to

$$\mathcal{P} - \frac{g}{\overline{\rho}}\,\overline{\rho' w'} - \epsilon = 0\,, \qquad \overline{\rho' w'}\,\frac{\partial \overline{\rho}}{\partial z} + \chi = 0\,.$$

In these equations, $\mathcal{P}$ is the turbulence production, ϵ is the dissipation rate, χ is the dissipation rate of mean-square density fluctuations, and overline denotes an appropriate horizontal spatial average.

(i) Define the flux Richardson number, Ri_f and the gradient Richardson number $Ri(z)$ for this flow.

(ii) Explain why it may be appropriate on dimensional grounds to parameterize ϵ and χ as

$$\epsilon = \frac{q^3}{L_u}\,, \qquad \chi = \frac{\overline{\rho'^2}\,q}{L_\rho}\,, \qquad q^2 = \overline{u'^2 + v'^2 + w'^2}\,,$$

where L_u and L_ρ are integral length scales of the velocity and density fluctuations respectively.

(iii) Assume that L_u and L_ρ remain in a fixed ratio irrespective of the overall stratification, and also that C_u and C_ρ are constants, where

$$C_u = \frac{-\overline{u'w'}}{q^2}\,, \qquad C_\rho^2 = \frac{\left(\overline{\rho' w'}\right)^2}{q^2\,\overline{\rho'^2}}\,.$$

Show that

$$C_u\, q^2\,\frac{\partial \overline{u}}{\partial z} = -\frac{g}{\overline{\rho}}\,C_\rho^2\,L_\rho q\,\frac{\partial \overline{\rho}}{\partial z} + \frac{q^3}{L_u}\,. \tag{1}$$

(iv) Define the quantity R as the ratio of the right hand side of equation (1) to the left hand side. Plot R against q for different values of the mean density gradient (including the limiting value of zero density gradient).

(v) Equilibrium (i.e. where equation (1) is satisfied) clearly corresponds to $R = 1$. For a given value of the density gradient, show that the minimum value of R always corresponds to $Ri_f = 1/2$.

(vi) By solving equation (1) for non-trivial values of q, show that

$$Ri_f = \frac{1}{2}\left[1 - \left(1 - 4\,\frac{L_\rho C_\rho^2}{L_u C_u^2}\,Ri\right)^{1/2}\right],$$

and thus identify a critical gradient Richardson number Ri_c.

(vii) Briefly discuss what this model implies for flows with $Ri > Ri_c$.

END OF PAPER