UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2009 9:00 am to 12:00 pm

PAPER 72

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

*Answer **THREE** questions out of questions **1 to 5** and*
***ONE** question out of questions **6 to 7**.*

*There are **SEVEN** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Carefully quoting all necessary results on finite element methods, identify a variational problem associated with the forced Airy equation

$$u'' - xu = 1, \qquad 0 \leqslant x \leqslant 1,$$

given with zero boundary conditions $u(0) = u(1) = 0$.

(b) Find explicitly the linear algebraic system that need be solved, once this Airy equation is discretized with the Ritz method, using piecewise-linear basis of chapeau functions.

2

Consider the multistep ODE method given by the polynomials

$$\rho(w) = w^2 - (1+a)\,w + a, \qquad \sigma(w) = \frac{1}{2}(1+a)\,w^2 + \frac{1}{2}(1-3a)\,w,$$

where a is a real parameter.

(a) What is the order of the method for different values of a? For which values of a is the method convergent?

(c) Identify all the values of a for which a convergent method of this kind is A-stable.

3

The advection equation $u_t = u_x$ is solved by the fully-discretized finite difference scheme

$$\begin{aligned} &\frac{1}{6}\mu(1+\mu)u^{n+1}_{m-1} + \frac{1}{3}(2-\mu)(1+\mu)u^{n+1}_{m} + \frac{1}{6}(2-\mu)(1-\mu)u^{n+1}_{m+1} \\ = \; &\frac{1}{3}(2-\mu)u^{n}_{m} + \frac{1}{3}(1+\mu)u^{n}_{m+1}\,. \end{aligned}$$

(a) Determine the order of the method.

(b) Find the range of Courant numbers μ for which the method is stable. You are allowed to choose the range within which your equation is solved and employ any technique of stability analysis, as long as all your steps are mathematically correct and you quote all the relevant theorems.

4

(a) Define what is meant by algebraic stability of a Runge–Kutta method.

(b) Formulate and prove Butcher's theorem on algebraically stable Runge–Kutta methods.

(c) Which of the following RK methods are algebraically stable?

1. $$\begin{array}{c|cc} \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\ \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array},$$

2. $$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} & 0 \\ \hline & \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{array},$$

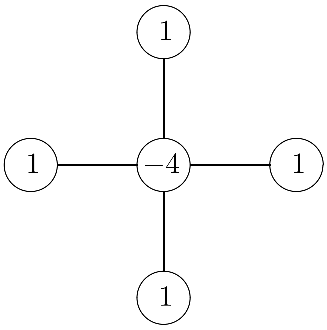
3. $$\begin{array}{c|cc} 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & \frac{5}{12} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}.$$

5

The Helmholtz equation

$$\nabla^2 u + \lambda u \;=\; 0\,,$$

given for $(x, y) \in [0, 1]$ with Dirichlet boundary conditions, is solved by the finite difference scheme

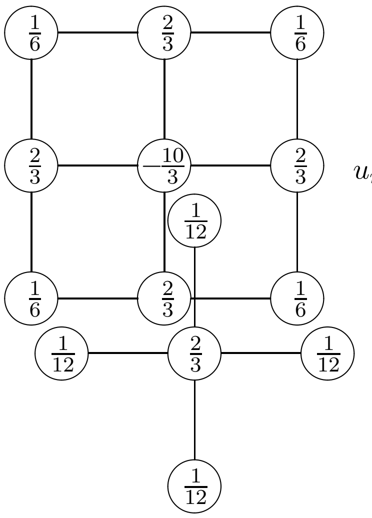


$$u_{m,n} + \lambda(\Delta x)^2 u_{m,n} \;=\; 0\,.$$

(a) Subject to which conditions on λ are the linear algebraic equations nonsingular?

(b) What is the local accuracy of this method?

(c) Consider now the method



$$u_{m,n} + \lambda(\Delta x)^2 \quad u_{m,n} \;=\; 0\,.$$

What is the local accuracy of the method?

6 Write an essay on the eigenvalue method for stability analysis of linear partial differential equations of evolution. You should describe the method and its scope and sketch its advantages and limitations. Illustrate your essay with examples.

7 Write an essay on multistep methods for ordinary differential equations, their definition, order, convergence and A-stability. State exactly all the theorems that you are quoting and illustrate your narrative with examples.

END OF PAPER