UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2009 9:00 am to 11:00 am

PAPER 71

A UNIFIED APPROACH TO BOUNDARY VALUE PROBLEMS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $q(x, y)$ satisfy the following initial-boundary value problem:

$$q_t + q_x + q_{xxx} = 0, \quad 0 < x < \infty, \quad t > 0,$$

$$q(x, 0) = q_0(x), \quad 0 < x < \infty,$$

$$q(0, t) = g_0(t), \quad t > 0,$$

where $q_0(x)$ has sufficient decay as $x \to \infty$, $q_0(x)$ and $g_0(t)$ have sufficient smoothness and $q_0(0) = g_0(0)$.

Assume that $q(x, t)$ has sufficient decay for all t as $x \to \infty$.

(a) Write the above PDE in the form

$$\left(e^{-ikx+\omega(k)t}q\right)_t + \left(e^{-ikx+\omega(k)t}X\right)_x = 0, \quad k \in \mathbb{C},$$

where $\omega(k)$ and $X(x, t, k)$ are to be determined.

(b) Find an integral representation for $q(x, t)$ in the complex k-plane involving appropriate transforms of $q_0(x)$, $g_0(t)$, $q_x(0, t)$ and $q_{xx}(0, t)$.

(c) Use the associated global relation to eliminate the transforms of $q_x(0, t)$ and $q_{xx}(0, t)$ and hence obtain an integral representation for $q(x, t)$ involving only transforms of $q_0(x)$ and $g_0(t)$.

(d) Prove that the expression obtained in (c) solves the above initial-boundary value problem.

2 Let Ω be the interior of a convex polygon in the complex z-plane with corners $z_1, ..., z_n$. Recall that if q satisfies Laplace's equation for $z \in \Omega$, then

$$q_z = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{l_j} e^{ikz} \hat{q}_j(k) dk, \quad z \in \Omega,$$

where

$$\hat{q}_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz} q_z dz, \quad j = 1, ..., n, \quad z_{n+1} = z_1,$$

and $\{l_j\}_1^n$ are the n-rays form the origin to infinity defined by

$$l_j = \{k \in \mathbb{C} : \arg k = -\arg(z_j - z_{j+1})\}, \quad j = 1, ..., n, \quad z_{n+1} = z_1.$$

Using the above result, find an integral representation for q_z in the case that Ω is the interior of the first quadrant of the complex z-plane and q satisfies the following boundary conditions:

$$-q_x(0, y) \sin \beta_1 + q_y(0, y) \cos \beta_1 = g_1(y), \quad 0 < y < \infty,$$

$$-q_y(x, 0) \sin \beta_2 + q_x(x, 0) \cos \beta_2 = g_2(x), \quad 0 < x < \infty,$$

where the complex-valued functions $g_1(y)$ and $g_2(x)$ have sufficient smoothness and decay and the real constants β_1, β_2 satisfy

$$0 \leqslant \beta_1 < \pi, \quad 0 \leqslant \beta_2 < \pi, \quad \beta_1 + \beta_2 = \frac{n\pi}{2}, \quad n = 0, 1, 2, 3.$$

Hint: You may find convenient to introduce the following unknown functions

$$u_1(y) = -q_y(0, y) \sin \beta_1 - q_x(0, y) \cos \beta_1, \quad 0 < y < \infty,$$

$$u_2(x) = q_x(x, 0) \sin \beta_2 + q_y(x, 0) \cos \beta_2, \quad 0 < y < \infty.$$

3 Obtain an integral representation for the solution of the following initial-boundary value problem:

$$q_t = q_{xx} + \alpha q_x, \quad 0 < x < \infty, \quad t > 0,$$

$$q(x, 0) = e^{-ax}, \quad 0 < x < \infty,$$

$$q(0, t) = \cos t, \quad t > 0,$$

where α and a are positive constants and $\alpha \neq a$.

Use appropriate contour deformation to obtain an integrand which decays exponentially for large k.

Suppose that $\exp(-ax)$ is replaced with $q_0(x)$, where $q_0(x)$ has sufficient decay. Can this problem be solved by an x-sine transform?

END OF PAPER