UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2009 1:30 pm to 3:30 pm

PAPER 70

NONLINEAR PATTERNS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A model of the earth's magnetic field is represented by the interaction of three modes, arising in a simple bifurcation, with the following symmetries:

(a) an axisymmetric dipole magnetic field; (b) an equatorial dipole magnetic field which changes sign under a rotation about the axis of π; (c) an induced velocity field with the same symmetry as the equatorial dipole.

It is assumed that the earth is symmetric with respect to longitude, and that the system is invariant under a change of sign of all magnetic fields, leaving the velocity field unchanged. Finally, it is known that the equations describing the magnetic field evolution are linear in the magnetic field.

Justify the statement that the axisymmetric field can be represented by a real variable B, while the other fields can be represented by complex variables A, V.

Write down the most general set of equations that respect the above conditions, and show that when these are truncated at cubic order they take the form, where the c_i, d_i are complex in general, and the μ_i, ω_i and e are real:

$$\dot{B} = \mu_1 B + (c_1 A^* V + c.c.) - e|V|^2 B,$$
$$\dot{A} = (\mu_2 + i\omega_2)A + c_2 VB - d_2|V|^2 A - d_3 V^2 A^*,$$
$$\dot{V} = (\mu_3 + i\omega_3)V + c_3 AB - d_4 V|V|^2 - d_5 V|A|^2 - d_6 V^* A^2 - d_7 VB^2,$$

and show that c_1 and $|c_3|$ can be taken as unity without loss of generality.

Now take $d_2 = d_3 = d_5 = d_6 = d_7 = 0$, and d_4, c_2 real. Derive equations (which need not be solved) governing the amplitudes and phases of the variables for solutions of the equations where the moduli of A, V, B are constant. Find explicitly a relation between the parameters for which such a solution with $A, B \neq 0$ bifurcates from one with $A, B = 0$, $V \neq 0$.

2 Consider the problem of steady-state bifurcation on a hexagonal lattice with symmetry under rotation through 60° but not under reflection. Using the natural two-dimensional representation, show that if the three fundamental wavenumbers of the lattice are $\mathbf{k_1} = k(1,0)$, $\mathbf{k}_{2,3} = k(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$, then the solution on the extended centre manifold can be written in the form

$$\sum_1^3 A_i(t)e^{i\mathbf{k}_i\cdot\mathbf{x}} \quad + \ c.c.,$$

where (identifying $A_{4,5}$ with $A_{1,2}$)

$$\frac{dA_i}{dt} = \mu A_i + \alpha A^*_{i+1}A^*_{i+2} - A_i(a|A_i|^2 + b|A_{i+1}|^2 + cA_{i+2}|^2) \ + \ \text{higher order terms},$$

and the coefficients are real.

Assume $\alpha > 0$ and neglect the higher order terms. Explain why, in this case, the only possible stable solutions lie in the subspace in which the A_i are all real and positive. Restricting attention to this subspace, and further assuming that $b+c > 2a > 0$, $(b-a)(c-a) < 0$, show (i) that there are steady solutions with either $A_1 \neq 0$, $A_2 = A_3 = 0$ (and cyclic permutations) [Rolls] or else $A_1 = A_2 = A_3$ [Hexagons], but none of 'rectangle' type, where e.g. $A_1 = A_2 \neq A_3$; (ii) that if μ is sufficiently large (so that the terms in α can be neglected in the stability calculation), there are no stable steady solutions of either of the above types.

Give a qualitative discussion of the dynamics in this case.

[*Hint: in the stability calculation for the hexagons, one eigenvector of the jacobian has all elements equal, while the other two eigenvectors have elements that sum to zero.*]

3 The analogue of the Landau-Ginzburg equation for the complex amplitude $A(X,T)$ for a mildly subcritical steady state bifurcation problem with $O(2)$ symmetry takes the form

$$\frac{\partial A}{\partial T} = \mu A + \alpha|A|^2 A - |A|^4 A + \frac{\partial^2 A}{\partial X^2},$$

where μ, α are real and $\alpha > 0$.

Find, for any given α and μ, (a) the range of existence and (b) the range of stability to long-wavelength disturbances of uniform steady solutions for which $A = Re^{iQX}$, with R, Q constant. For fixed α, sketch the regions of (μ, R) space and (μ, Q) space in which such stable solutions can be found.

END OF PAPER