UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2009 1:30 pm to 4:30 pm

PAPER 7

TOPICS IN FOURIER ANALYSIS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) If $A > 0$ and N is a positive integer, define $K_{A,N} : \mathbb{T}^2 \to \mathbb{R}$ by

$$K_{A,N}(s,t) = A(2 + \cos s + \cos t)^N.$$

Show that, given any $\epsilon > 0$, we can find $A > 0$ and N such that

$$\begin{aligned} K_{A,N}(s,t) &\geqslant 0 && \text{for all } (s,t), \\ |K_{A,N}(s,t)| &\leqslant \epsilon && \text{whenever } |s| \geqslant \epsilon, \\ |K_{A,N}(s,t)| &\leqslant \epsilon && \text{whenever } |t| \geqslant \epsilon, \end{aligned}$$

and

$$\frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} K_{A,N}(s,t)\, ds\, dt = 1.$$

By considering functions of the form

$$P(x,y) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} K_{A,N}(x-s, y-t) f(s,t)\, ds\, dt$$

show that any continuous function $f : \mathbb{T}^2 \to \mathbb{R}$ can be uniformly approximated by trigonometric polynomials.

(ii) Write $|E|$ for the number of elements in a finite set E. State and prove a necessary and sufficient condition on a point $(u,v) \in \mathbb{T}^2$ for the following result to be true:

$$n^{-1} \left|\{1 \leqslant r \leqslant n \,:\, (ru, rv) \in [a,b) \times [c,d)\}\right| \to (2\pi)^{-2}(b-a)(d-c)$$

as $n \to \infty$ for all $0 \leqslant a \leqslant b \leqslant 2\pi$ and $0 \leqslant c \leqslant d \leqslant 2\pi$.

2 (a) State Minkowski's Fundamental Theorem for the geometry of numbers and use it to show that if x is real there exist infinitely many pairs of integers n and m with $n \neq 0$ such that

$$\left| x - \frac{m}{n} \right| \leqslant \frac{1}{n^2}.$$

(b) Develop the theory of Fourier Analysis on finite Abelian groups up to and including the identification of $\hat{\hat{G}}$ with G and the inversion theorem. [*If you use results like the Structure Theorem for finite Abelian groups you must prove them.*]

3 (a) Suppose that $\beta : [1, \infty) \to \mathbb{R}$ is an increasing function. Show that, if

$$\int_1^X \frac{\beta(x) - x}{x^2}\, dx$$

tends to limit as $X \to \infty$, then $x^{-1}\beta(x) \to 1$ as $x \to \infty$.

(b) Suppose that Ω is an open set in $\mathbb{C}$ with $\Omega \supseteq \{z : \Re z \geqslant 0\}$. Let $F : \Omega \to \mathbb{C}$ be an analytic function and $f : [0, \infty] \to \mathbb{R}$ a bounded locally integrable function. If

$$F(z) = \int_0^\infty f(t) e^{-tz}\, dt$$

for $\Re z > 0$, show that $\int_0^\infty f(t)\, dt$ converges.

4 (i) Suppose that $A_r \to 0$ as $r \to \infty$. Show that

$$\left| \sum_{r=1}^{n} A_r \left(\frac{\sin rk}{rk} \right)^2 \right| \leqslant n \sup_r |A_r|,$$

that $\sum_{r=n+1}^{\infty} A_r \left(\frac{\sin rk}{rk} \right)^2$ converges absolutely and that

$$\left| \sum_{r=n+1}^{\infty} A_r \left(\frac{\sin rk}{rk} \right)^2 \right| \leqslant 2n^{-1} \sup_{r \geqslant n+1} |A_r|.$$

Hence show that

$$k \sum_{r=1}^{n} A_r \left(\frac{\sin rk}{rk} \right)^2 \to 0$$

as $k \to 0$.

[*You may use the estimate* $\sum_{r=n+1}^{\infty} r^{-2} \leqslant 2n^{-1}$ *without proof.*]

If $a < c < b$, $F : [a, b] \to \mathbb{C}$ is continuous, $F(t) = At + B$ for $t \in [a, c]$, $F(t) = A't + B'$ for $t \in [c, b]$, and

$$\frac{F(t+h) - 2F(t) + F(t-h)}{h} \to 0$$

as $h \to 0$, show that $A = A'$, $B = B'$.

(ii) Let E be a finite set. Suppose that

$$\sum_{r=-n}^{n} a_r \exp(irt) \to 0$$

as $n \to \infty$ for all $t \in \mathbb{T} \setminus E$. Show that $a_r = 0$ for all r. [*You may find part (i) useful in showing that a certain piecewise linear function is, in fact, linear.*]

END OF PAPER