UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2009 9:00 am to 12:00 pm

PAPER 69

APPROXIMATION THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let σ_n be the Fejer operator, i.e., for a 2π-periodic function $f \in C(\mathbb{T})$,

$$\sigma_n(f,x) = \int_{-\pi}^{\pi} f(x-t)F_n(t)\,dt, \qquad F_n(t) := \frac{1}{\pi}\frac{1}{2n}\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}, \qquad \int_{-\pi}^{\pi} F_n(t)dt = 1.$$

Prove the estimate

$$\|\sigma_n(f) - f\|_\infty \leqslant c\,\omega(f,\delta_n), \qquad \delta_n = \tfrac{\ln n}{n}\,,$$

where $\omega(f,\delta)$ is the modulus of continuity of f. Hence derive that, for any continuous function f, the Fejer sums $\sigma_n(f)$ converge uniformly to f.

2

For $f \in C[0,1]$, write down the definition of the Bernstein polynomial $B_n(f)$, and derive expression for the first and, consequently, for the r-th derivative $B_n^{(r)}(f,x)$.

Show that $B_n^{(r)}(f,x) = B_{n-r}(g_r,x)$ with a certain relation between g_r and f. Hence derive that

$$B_n^{(r)}(f) \to f^{(r)}\,,$$

i.e., that $B_n^{(r)}(f)$ converge uniformly to $f^{(r)}$.

Hint. You may use without the proof that, for $f \in C^r[0,1]$, uniformly in x,

$$\lim_{h\to 0} h^{-r}\Delta_h^r f(x) = f^{(r)}(x)\,,$$

where $\Delta_h^1 f(x) = f(x+h) - f(x)$ and $\Delta_h^r f(x) = \Delta_h^1(\Delta_h^{r-1} f(x))$.

3

1) Let $\mathcal{S}_k(\Delta)$ be the space of splines of degree $k-1$ spanned by the B-splines $(N_j)_{j=1}^n$ on a knot sequence $\Delta = (t_j)_{j=1}^{n+k}$ such that $t_j < t_{j+k}$. Let $x = (x_i)_{i=1}^n$ be interpolation points obeying the conditions

$$N_i(x_i) > 0\,,$$

and let $P_x : C[a,b] \to \mathcal{S}_k(\Delta)$ be the map which associates with any $f \in C[a,b]$ the spline $P_x(f)$ from $\mathcal{S}_k$ which interpolates f at (x_i). Prove that

$$\|P_x\|_{L_\infty} \leqslant \|A_x^{-1}\|_{\ell_\infty}$$

where A_x is the matrix $(N_j(x_i))_{i,j=1}^n$.

2) Consider the case of quadratic interpolating splines on the uniform knot-sequence $(t_1, t_2, \ldots, t_{n+3}) = (1, 2, \ldots, n+3)$ with the interpolating points

$$x_i = \frac{1}{2}(t_i + t_{i+3}) = i + 3/2, \quad i = 1, \ldots, n\,.$$

a) Using the recurrence relation between linear and quadratic B-splines, or otherwise, determine the values of N_j at the points (x_i).

b) Write down the matrix $A_x = (N_j(x_i))$, and evaluate the norm $\|A^{-1}\|_{\ell_\infty}$. (You may use any appropriate theorem on the inverse of certain matrices if correctly stated).

c) Hence show that $\|P_x\|_{L_\infty} \leqslant 2$.

4

a) State the Kolmogorov criterion for the element of best approximation to a real-valued function $f \in C[0,1]$ from a linear subspace $\mathcal{U}$ of $C[0,1]$.

b) From this criterion, derive the Chebyshev alternation theorem for the element of best approximation to a function $f \in C[0,1]$ from $\mathcal{P}_n$, the space of all algebraic polynomials of degree n.

5 Given a knot sequence $\Delta = (t_i)_{i=1}^{n+k}$, let ω_i and $\ell_i(\cdot, t)$ be polynomials in $\mathcal{P}_{k-1}$ defined by

1) $\omega_i(x) := (x - t_{i+1}) \cdots (x - t_{i+k-1})$,
2) $\ell_i(\cdot, t)$ interpolates $(\cdot - t)_+^{k-1}$ on $x = t_i, ..., t_{i+k-1}$.

Further, let

$$N_i := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}$$

be the B-spline of order k with the knots $t_i, \ldots, t_{i+k}$.

a) Prove Lee's formula

$$\omega_i(x) N_i(t) = \ell_{i+1}(x, t) - \ell_i(x, t), \qquad \forall x, t \in \mathbb{R},$$

and derive from it the Marsden identity:

$$(x - t)^{k-1} = \sum_{i=1}^{n} \omega_i(x) N_i(t), \quad t_k < t < t_{n+1}, \quad \forall x \in \mathbb{R}.$$

b) From the Marsden identity, find the coefficients $a_i^{(m)}$ in the B-spline representation of monomials t^m:

$$t^m = \sum_{i=1}^{n} a_i^{(m)} N_i(t), \quad t_k < t < t_{n+1}, \quad \text{for} \quad m = 0, \ldots, k-1.$$

6 Let $E_n(f)$ be the value of the best approximation of a 2π-periodic f by trigonometric polynomials of degree n, and let $\omega(f, \delta)$ be the modulus of continuity of f.

a) State the inverse theorem for trigonometric approximation and show that

$$E_n(f) = \mathcal{O}(n^{-\alpha}) \quad \text{implies} \quad \omega(f, \tfrac{1}{n}) = \begin{cases} \mathcal{O}(n^{-\alpha}), & 0 < \alpha < 1, \\ \mathcal{O}(\frac{\ln n}{n}), & \alpha = 1\,. \end{cases}$$

b) Find the order of $\omega(g, \delta)$ for the Weierstrass functions

$$g(x) := \sum_{k=0}^{\infty} \frac{1}{a^k} \cos 5^k x, \qquad 1 < a \leqslant 5,$$

using the fact that, for $5^m \leqslant n < 5^{m+1}$, the polynomial of best approximation of degree n to g is the partial sum $t_n(x) = \sum_{k=0}^{m} \frac{1}{a^k} \cos 5^k x$.

c) For the case $a = 5$, show that

$$|g(x + \tfrac{1}{n}) - g(x)| \geqslant \frac{c \ln n}{n}, \qquad x = \frac{\pi}{2},$$

hence derive that

$$\omega(g, \tfrac{1}{n}) \geqslant \frac{c \ln n}{n}.$$

Explain briefly why the class of functions with $E_n(f) = \mathcal{O}(\frac{1}{n})$ cannot be characterized in terms of $\omega(f, \frac{1}{n})$.

END OF PAPER