UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2009 9:00 am to 11:00 am

PAPER 68

ACCRETION DISCS

*There are **THREE** questions in total.*

*Full marks can be obtained by completing **TWO** questions.*

The questions carry equal weight.

*This is an **OPEN BOOK** examination.*
Candidates may bring handwritten notes and lecture handouts into the examination.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A steady accretion disc, around a star of mass M, loses mass to a wind at a rate $S(R)$ per unit area per unit time. If $\dot{M}(R)$ is the rate at which mass flows inwards at radius R, show that

$$\frac{d\dot{M}}{dR} = 2\pi R S\,.$$

At radii $R > R_0$, there is no wind so that $S = 0$ and $\dot{M} = \dot{M}_0 = \text{const.}$ At $R < R_0$, the energy put into driving the wind is a fraction f $(0 < f \leqslant 1)$ of the energy dissipated locally by the viscosity ν. The energy (per unit mass) given to the wind is such that when it leaves the disc surface its net binding energy is zero (i.e. the wind is launched at the local escape speed). Show that this implies

$$\nu\Sigma = \frac{1}{9\pi f} R \frac{d\dot{M}}{dR}\,,$$

where Σ is the surface density.

The specific angular momentum carried away by the wind is $h = R^2\Omega$ where Ω is the angular velocity. Show that conservation of angular momentum then implies

$$R^{1/2}\frac{d}{dR}\left(R^{3/2}\frac{d\dot{M}}{dR}\right) = \frac{3}{2} f \dot{M}\,.$$

The viscous torque can be assumed to vanish at the inner disc radius $R = R_*$. For the case $f = \frac{1}{3}$, find $\dot{M}(R)$.

If $R_* \ll R_0$, show that

$$\frac{\dot{M}\left(R_*\right)}{\dot{M}_0} = \frac{3}{2}\left(\frac{R_*}{R_0}\right)^{\frac{1}{2}}\,.$$

2 A star of mass M, radius R_* and luminosity L_* is accreting steadily from a disc at rate $\dot{M}$. Explain what is meant by the disc being a passive disc. Give a rough criterion for the disc to be passive.

At large radii, $R \gg R_*$, a passive disc has a surface at $z_s(R)$, where $R_* \ll z_s \ll R$. The central star may be treated as an isotropic point source of radiation. Show that at radius R the flux of radiation received by the disc surface is approximately

$$F_d(R) = \frac{L_*}{4\pi R}\frac{d}{dR}\left(\frac{z_s}{R}\right) .$$

Assume that the disc is locally isothermal with temperature $T_c(R)$ given by $F_d(R) = \sigma T_c^4$, where σ is the Stefan-Boltzmann constant. Assume also that the disc is supported by gas pressure, $p = (\mathcal{R}/\mu)\rho\mathcal{T}$,where $\mathcal{R}$ is the gas constant, μ is the mean molecular mass and ρ is the density, and that $z_s = fH$, where $H(R)$ is the disc scale height and f is a constant of order unity. Find a first-order differential equation which relates $q \equiv z_s(R)/R$ and R.

If the disc becomes physically thick, i.e. $z_s \to R$ at some large radius $R_M \gg R_*$, show that for intermediate radii, $R_* \ll R \ll R_M$,

$$\frac{z_s}{R} = \left(\frac{R}{R_0}\right)^{\frac{2}{7}} \qquad (*)$$

where

$$R_0^2 = \left(\frac{14\pi\sigma}{L_*}\right)\left(\frac{1}{f^8}\right)\left(\frac{GM}{\mathcal{R}/\mu}\right)^4 .$$

The stellar surface scale height H_* is given by

$$H_*^2 = \frac{\mathcal{R}\mathcal{T}_*/\mu}{GM/R_*^3} ,$$

where T_* is the stellar surface temperature. In general $fH_* \ll R_*$. Show that this implies $R_0 \gg R_*$ and hence that the relationship $(*)$ has some radial range of validity.

3 The equations governing the radial propagation of axisymmetric ($m = 0$) waves in a thin, non-viscous Keplerian disc are

$$\frac{d\xi}{dz} - \frac{N^2}{g}\xi - \left(\omega^2 - N^2\right) u_z = 0\,,$$

and

$$\frac{du_z}{dz} - \frac{\rho g}{\gamma p}u_z + \left(\frac{\rho}{\gamma p} - \frac{k^2}{\omega^2 - \Omega_0^2}\right)\xi = 0\,.$$

Here u_z is the vertical component of velocity and $\xi = i\omega p'/\rho$ is a measure of the pressure perturbation p'. The waves have frequency ω and radial wavenumber k. The perturbations are adiabatic, so that $p' = (\gamma p/\rho)\rho'$, $g(> 0)$ is the vertical gravity, Ω_0 the local orbital frequency and N the Brunt-Väisälä frequency. If the disc is locally isothermal, with (isothermal) sound speed c_s independent of z, show that

$$\rho(R, z) = \rho(R, 0)e^{-z^2/2H^2}\,,$$

where $H = c_s/\Omega_0$.

Show also that in this case

$$N^2 = \left(1 - \frac{1}{\gamma}\right)\left(\frac{z^2}{H^2}\right)\Omega_0^2\,.$$

In the case where the perturbations are isothermal, so that $\gamma = 1$, show that the equations may be written in the form

$$\frac{d^2\xi}{dx^2} - x\frac{d\xi}{dx} + \alpha\xi = 0\,, \qquad (*)$$

where $x = z/H$ and

$$\alpha = \frac{\omega^2}{\Omega_0^2}\left[1 - \frac{k^2 c_s^2}{\omega^2 - \Omega_0^2}\right]\,.$$

State without proof why we require solutions of $(*)$ for which ξ is a polynomial. Show that for such solutions we require that $\alpha = n$ for $n = 0, 1, 2, \ldots$.

Give the dispersion relation explicitly in the form $\omega^2 = f(kH)$ for the cases $n = 0$ and $n = 1$, and describe the nature of the waves in each case.

For the case $n = 1$, write down the generalisation of the dispersion relation to the non-axisymmetric waves with azimuthal wavenumber $m = 1$.

Show that for these waves in the low frequency ($\omega \ll \Omega_0$), long-wavelength ($kH \ll 1$) limit the dispersion relation is

$$\frac{\omega}{\Omega_0} = \pm\tfrac{1}{2}kH\,.$$

What do these waves represent?

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