UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2009 1:30 pm to 4:30 pm

PAPER 67

STELLAR AND PLANETARY MAGNETIC FIELDS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A two-dimensional magnetic field $\mathbf{B}(x, z, t)$ permeates all space. It is acted on by a solenoidal two-dimensional velocity field $\mathbf{u}(x, z, t)$.

Show that the magnetic vector potential can be written in the form $\mathbf{A} = (0, B_0 z + a, 0)$ where a satisfies:

$$\dot{a} + \mathbf{u} \cdot \nabla a + wB_0 = \eta \nabla^2 a \tag{1}$$

where $\mathbf{B_0} = (B_0, 0, 0)$ is a uniform magnetic field of strength B_0, η is the magnetic diffusivity and $w = \mathbf{u} \cdot \hat{\mathbf{z}}$.

Suppose that there is no mean flow in the z-direction and suppose that all relevant averages in x exist, and are independent of time. In addition it is given that a and $\mathbf{u}$ both vanish at infinity. Using overbar $\bar{\cdot}$ to denote such average, derive the expressions

$$\overline{wa'} = \eta \frac{d\bar{a}}{dz} + \langle wa' \rangle, \tag{2}$$

$$\left\langle \overline{wa'} \frac{d\bar{a}}{dz} \right\rangle + \langle wa' \rangle B_0 = -\eta \langle |\nabla a'|^2 \rangle, \tag{3}$$

where $a = \bar{a} + a'$ and $\langle . \rangle$ denotes an average over all space. Show that the last equation reduces to Zeldovich's result

$$\langle wa' \rangle B_0 = -\eta \langle |\nabla a'|^2 \rangle - \eta \left\langle \left(\frac{d\bar{a}}{dz} \right)^2 \right\rangle. \tag{4}$$

Now suppose that $\overline{wa'} = -\eta_1 B_0 f(z)$, where $\langle f(z) \rangle = 1$ and η_1 is a turbulent magnetic diffusivity. Show that

$$\frac{\eta}{\eta_1} > \Big(\langle f^2 \rangle - 1 \Big). \tag{5}$$

Explain what form must be taken by f if $\eta_1 \gg \eta$. It is now given that a is periodic in z with zero mean and period d. Show by using the Schwarz inequality and the result that

$$\langle |\nabla a'|^2 \rangle \geqslant \frac{\pi^2}{d^2} \langle a'^2 \rangle \tag{6}$$

that

$$\eta_1^2 [\langle f^2 \rangle - 1] \leqslant \langle w^2 \rangle \frac{d^2}{4\pi^2}. \tag{7}$$

2 Small scale magnetic and solenoidal velocity fields **b** and **u** are induced by a steady applied force **f** and a source of magnetic flux **g**, in the presence of a mean magnetic field **B**. It may be assumed that first-order smoothing is applied. Justify the equations

$$0 = -\frac{1}{\rho}\nabla P + \frac{1}{\mu_0\rho}\mathbf{B}.\nabla\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{f}, \qquad (1)$$

$$0 = \mathbf{B}.\nabla\mathbf{u} + \eta\nabla^2\mathbf{b} + \mathbf{g}, \qquad (2)$$

$$\nabla\cdot\mathbf{u} = 0. \qquad (3)$$

where η is the magnetic diffusivity and ν is the kinematic viscosity.

Calculate the mean electromotive force $\mathcal{E} = \overline{\mathbf{u}\times\mathbf{b}}$ under the assumptions that **f** and **g** are solenoidal and monochromatic, so that $\nabla^2\mathbf{f} = -k^2\mathbf{f}$, $\nabla^2\mathbf{g} = -k^2\mathbf{g}$, and that $\mathcal{E} = 0$ when $|\mathbf{B}| = 0$. Relate the result you have obtained to the magnetic and kinetic helicities of the flow. Writing

$$\mathcal{E}_i = \alpha_{ij}(|\mathbf{B}|)B_j, \qquad (4)$$

and assuming that the statistics of **f** and **g** are isotropic, show that $\alpha_{ij} \sim |B|^{-3}$ as $|B| \to \infty$. (Detailed calculation is not required.)

3 Write down the dimensionless equation governing Boussinesq thermal convection in an imposed vertical magnetic field, defining the Rayleigh number R, the Chandresekhar number Q and the diffusivity ratios σ and ζ in terms of dimensional parameters. Derive the dispersion relation for the complex growth rate s,

$$\beta^2(s+\beta^2)(s+\sigma\beta^2)(s+\zeta\beta^2) + \sigma Q\beta^2\pi^2(s+\beta^2) - R\sigma k^2(s+\zeta\beta^2) = 0, \qquad (1)$$

where $\beta^2 = k^2 + \pi^2$ and k is the horizontal wavenumber. Derive the relation governing the onset of steady convection and oscillatory convection, and give the condition on the parameters that ensures that the steady and oscillatory solution branches coincide. Give approximate expressions, valid in the limit of large Q, for the two situations in which: (a) the branches merge at the minimum (in R, k space) of the steady solution branch;(b) they merge at the minimum of the oscillatory branch.
[HINT: when $Q \gg 1$, the critical wavenumber for both the steady and oscillatory convection is $O(Q^{1/6})$.]

4 Write an essay on planetary magnetic fields. You should give a discussion of relevant observations for all planets in the solar system and how they have led to the conclusion that dynamo mechanisms currently operate in some planets but not others, indicating how planetary magnetic fields may be maintained.

END OF PAPER