UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2009 1:30 pm to 4:30 pm

PAPER 66

PHYSICAL COSMOLOGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) From the two Friedmann equations,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} \qquad (\dagger)$$

and

$$\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G p}{c^2} + \Lambda c^2 \qquad (\ddagger)$$

where the dot denotes differentiation with respect to time, a is the scale factor, k is the curvature parameter, Λ is the cosmological constant, and the other symbols have their usual meanings, verify that the equation linking the mass density ρ and pressure p may be written in the form:

$$\frac{d}{dt}(\rho a^3) + \frac{p}{c^2}\frac{d}{dt}(a^3) = 0$$

(ii) Assuming that $p = w\rho c^2$, where w is a constant, show that:

$$\rho a^{3(1+w)} = C \qquad (*)$$

where C is a constant.

(iii) Consider the case $w = -1$. By referring to equations ($\dagger$) and ($\ddagger$), explain why a fluid with such an equation of state is dynamically indistinguishable from the effect of a cosmological constant.

(iv) Consider a Universe with $\Lambda = 0$ and $k = 0$, and total energy content $\rho = \rho_{\rm M} + \rho_{\rm DE}$, where $\rho_{\rm M}$ and $\rho_{\rm DE}$ evolve independently according to equation ($*$), with $w_{\rm M} = 0$ and $w_{\rm DE} = w$ respectively. Show that in such a Universe the Hubble parameter $H(a)$ obeys the equation:

$$H(a)^2 = H_0^2\left[\Omega_{\rm DE,0}\, a^{-3}\left(a^{-3w} - 1\right) + a^{-3}\right]$$

where the subscript 0 denotes the present time and $a_0 = 1$. Define $\Omega_{\rm DE,0}$ in your answer.

(v) Discuss the behaviour of $H(a)$ and $\rho_{\rm DE}/\rho_{\rm M}$ as $a \to 0$, *for the case* $w < 0$. Does the expansion rate at early times depend on the value of $\Omega_{\rm DE,0}$? Comment on your answer.

2 (i) Show that for a pressureless matter-dominated Universe with $\Lambda = 0$,

$$\frac{kc^2}{a_0^2} = (2q_0 - 1)H_0^2,$$

and

$$\rho_0 = \frac{3H_0^2}{4\pi G} q_0,$$

where $q \equiv -\frac{a\ddot{a}}{\dot{a}^2}$ is the deceleration parameter, a is the scale factor, k is the curvature, H is the Hubble parameter, ρ is the density, the subscript 0 indicates the present time, and the other symbols have their usual meanings.

(ii) Show that the age of the Universe, t_0, in such a model is given by:

$$t_0 = \frac{1}{H_0} \int_0^1 \left[1 - 2q_0 + \frac{2q_0}{x}\right]^{-1/2} dx$$

where $x = a/a_0$

(iii) By making the substitution $1 - \cos\theta = \left(\frac{2q_0 - 1}{q_0}\right) x$, show that the integral in part (ii) can be written as:

$$H_0 t_0 = \frac{q_0}{(2q_0 - 1)^{3/2}} \int_0^{\theta_*} \frac{\sin\theta \sqrt{1 - \cos\theta}}{\sqrt{1 + \cos\theta}} \, d\theta$$

where $(1 - \cos\theta_*) = \left(\frac{2q_0 - 1}{q_0}\right)$.

(iv) From the evolution of the Hubble parameter with redshift z:

$$H(z) = H_0 \left(\Omega_{\mathrm{r},0} \cdot (1+z)^4 + \Omega_{\mathrm{m},0} \cdot (1+z)^3 + \Omega_{\mathrm{k},0} \cdot (1+z)^2 + \Omega_{\Lambda,0}\right)^{1/2},$$

where $\Omega_{\mathrm{r},0}$, $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{k},0}$, and $\Omega_{\Lambda,0}$ are the present-day contributions to the critical density by, respectively, radiation, matter, curvature and the cosmological constant, derive an expression for $H(z)$ in an empty Universe with $\Lambda = 0$. What is the value of q_0 in such a Universe? Explain the apparent inconsistency between your two answers to this part of the question.

3 (i) Describe in a few sentences what is meant by the 'Lyman alpha forest'.

(ii) Show with the aid of simple physical arguments why the Lyman alpha forest is thought to trace gas at, or near, the cosmic mean density.

(iii) Consider two Lyman alpha forest clouds detected at redshifts z_1 and z_2 ($z_2 > z_1$) along the line of sight to a quasar. Derive an expression relating the redshift difference $\Delta z = z_2 - z_1$ to the distance between the two clouds when $\Delta z \ll 1$.

(iv) Describe what is meant by the 'proximity effect' in the spectra of distant quasars. Explain how the proximity effect can be used to estimate the intensity of the intergalactic hydrogen ionising background, and comment on the uncertainties involved in this estimate.

(v) Summarise current ideas on the origin of the ionising background and its evolution with redshift.

4

(i) Explain what is meant by the "*Comoving particle horizon*".
From the Robertson-Walker metric:

$$(ds)^2 = (c\,dt)^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]$$

and the evolution of the Hubble parameter with the scale factor a:

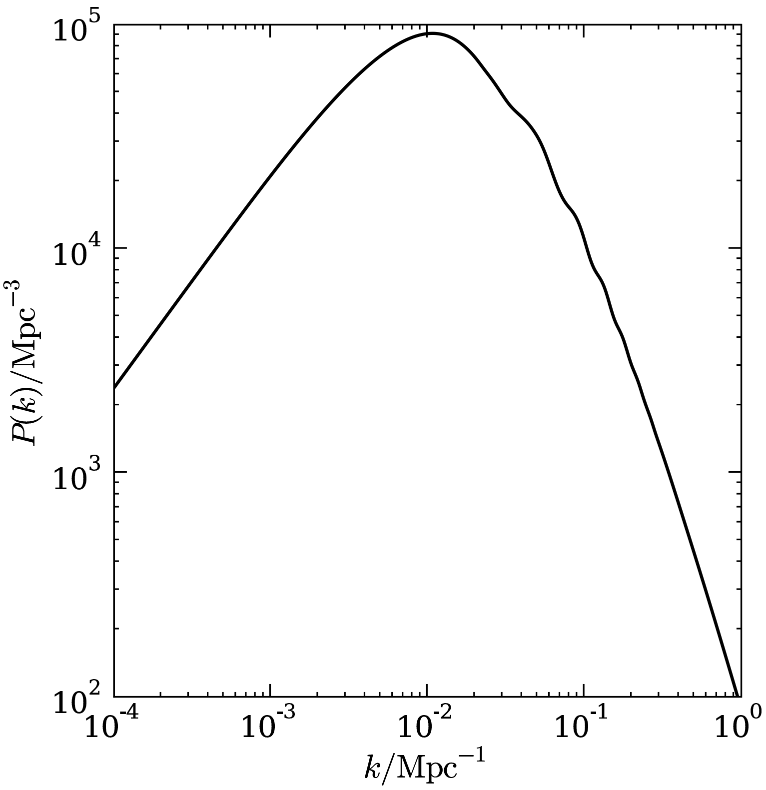
$$H(a) = H_0\left(\Omega_{\mathrm{r},0}\cdot a^{-4} + \Omega_{\mathrm{m},0}\cdot a^{-3} + \Omega_{\mathrm{k},0}\cdot a^{-2} + \Omega_{\Lambda,0}\right)^{1/2} ,$$

where $\Omega_{\mathrm{r},0}$, $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{k},0}$, and $\Omega_{\Lambda,0}$ are the present-day contributions to the critical density by, respectively, radiation, matter, curvature and the cosmological constant, show that, in a flat Universe, the comoving particle horizon r at redshift z is given by:

$$r = \frac{c}{H_0}\int_0^{(1+z)^{-1}} \left(\Omega_{\mathrm{r},0} + \Omega_{\mathrm{m},0}\cdot a + \Omega_{\Lambda,0}\cdot a^4\right)^{-1/2} da \qquad (*)$$

(ii) Simplify and solve the integral $(*)$ for the Universe prior to matter-radiation equality ($z \gg z_{\mathrm{eq}} \simeq 3000$). Using the expression you have derived, estimate the comoving particle horizon in megaparsecs at matter-radiation equality. Comment on the validity of your answer. Use similar arguments to estimate the particle horizon today.

(iii) Recent galaxy surveys are estimated to be complete (in a given patch of sky) for V magnitudes $m_{\mathrm{V}} \leqslant 17$. Assuming that the Milky Way, which has an absolute magnitude $M_{\mathrm{V}} = -21$, is a typical galaxy, estimate the distance in Mpc to which such surveys probe typical galaxies. (You may ignore the K-correction, and assume that the distances involved are sufficiently small for $d_{\mathrm{P}} \simeq d_{\mathrm{L}} \simeq d_{\mathrm{A}}$, where $d_{\mathrm{P}}, d_{\mathrm{L}}$, and d_{A} are, respectively, the proper, luminosity, and angular diameter distances.)



(iv) The Figure shows the power spectrum of the large-scale distribution of matter. Based on your answers to parts (ii) and (iii) above, indicate on a sketch of the Figure (or state in writing):

(a) the scale (i.e. the value of k) of the horizon at the epoch of matter-radiation equality;

(b) the range of scales over which you expect Cosmic Microwave Background experiments to be useful in determining the matter power spectrum, given that the best-resolution experiments probe fluctuations on angular scales of $\sim 1 \times 10^{-3}$ radians;

(c) the range of scales over which you expect galaxy surveys to be useful in constraining the power spectrum.

Give a brief explanation for the behaviour of $P(k)$ around the value of k at point (a), and give reasons for your answers to points (b) and (c). Discuss the relative merits of methods (b) and (c) for an observational cosmologist keen to determine the matter power spectrum as accurately as possible.

[You may assume $\Omega_{\mathrm{m},0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$ *and* $c/H_0 = 4000\,\mathrm{Mpc}$, *where* c *is the speed of light and* H_0 *is the present-day value of the Hubble parameter.]*

END OF PAPER