UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2009 1:30 pm to 4:30 pm

PAPER 64

ASTROPHYSICAL FLUID DYNAMICS

*Attempt no more than **THREE** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + u\cdot\nabla\rho &= -\rho\nabla\cdot u\,,\\
\frac{\partial p}{\partial t} + u\cdot\nabla p &= -\gamma p\nabla\cdot u\,,\\
\rho\left(\frac{\partial u}{\partial t} + u\cdot\nabla u\right) &= -\rho\nabla\Phi - \nabla p + \frac{1}{\mu_0}(\nabla\times B)\times B\,,\\
\frac{\partial B}{\partial t} &= \nabla\times(u\times B)\,,\\
\nabla\cdot B &= 0\,,\\
\nabla^2\Phi &= 4\pi G\rho\,.
\end{aligned}$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Discuss the behaviour of the 'pre-Maxwell' equations

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E} \,,$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \,,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

under a Galilean transformation to a frame of reference moving with uniform relative velocity $\boldsymbol{v}$. Hence derive the ideal induction equation governing the evolution of the magnetic field in a conducting fluid. State briefly what conditions are required for its validity. State without proof the implications of the ideal induction equation for the evolution of the magnetic field lines and the magnetic flux.

(b) The magnetic helicity in a volume V is

$$H_{\mathrm{m}} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V \,,$$

where $\boldsymbol{A}$ is the magnetic vector potential. Derive an expression for the Lagrangian derivative of $\boldsymbol{A} \cdot \boldsymbol{B}/\rho$. Hence, or otherwise, show that the magnetic helicity in a material volume V is both uniquely defined and also conserved, provided that the magnetic field lines do not penetrate the boundary of V.

(c) A magnetic field is defined in cylindrical polar coordinates (R, ϕ, z) by

$$\boldsymbol{B} = \begin{cases} aR\,\boldsymbol{e}_\phi + b\,\boldsymbol{e}_z \,, & R < R_0 \,, \\ \boldsymbol{0} \,, & R > R_0 \,, \end{cases}$$

where a, b and R_0 are constants. Explain why this represents a twisted magnetic flux tube. Find the most general form of the magnetic vector potential and use this to calculate the magnetic helicity of the tube per unit length. Discuss whether the answer is unique.

2

(a) An isothermal gas undergoes a steady one-dimensional flow in the x-direction in a gravitational potential $\Phi(x)$. Show that the Mach number $\mathcal{M}$ satisfies

$$\frac{1}{2}\mathcal{M}^2 - \ln\mathcal{M} + \frac{\Phi}{c_{\mathrm{s}}^2} = \text{constant}\,,$$

where c_{s} is the isothermal sound speed. Deduce that the flow can make a sonic transition at a maximum of the potential, and determine the value of the constant in this case.

(b) Show that, in a steady axisymmetric flow in ideal MHD,

(i) the poloidal velocity is parallel to the poloidal magnetic field;

(ii) the angular velocity is approximately constant along each magnetic field line when the poloidal flow is highly sub-Alfvénic.

(c) Consider a thin accretion disc in the gravitational potential of a point mass M. The disc has Keplerian angular velocity $\Omega = (GM/R^3)^{1/2}$, where (R, ϕ, z) are cylindrical polar coordinates. The poloidal magnetic field lines just above the surface of the disc may be approximated as straight lines inclined at an angle i to the vertical. Describe the variation of the centrifugal–gravitational potential along such a field line, for matter that corotates with its footpoint $(R, z) = (R_0, 0)$, by expanding the potential to second order in the distance from the footpoint (or otherwise). Deduce that an outflow is accelerated from the surface of the disc if $i > 30°$.

3 An ideal gas of adiabatic exponent γ flows in one dimension in the absence of boundaries, gravity and magnetic fields.

(a) Determine all possible smooth local solutions of the equations of one-dimensional gas dynamics that depend only on the variable $\xi = x/t$ for $t > 0$. Show that one such solution is a rarefaction wave in which $\mathrm{d}u/\mathrm{d}\xi = 2/(\gamma+1)$. How do the adiabatic sound speed and specific entropy vary with ξ?

(b) At $t = 0$ the gas is initialized with density

$$\rho = \begin{cases} \rho_\mathrm{L}\,, & x < 0\,, \\ \rho_\mathrm{R}\,, & x > 0\,, \end{cases}$$

pressure

$$p = \begin{cases} p_\mathrm{L}\,, & x < 0\,, \\ p_\mathrm{R}\,, & x > 0\,, \end{cases}$$

and velocity

$$u = \begin{cases} u_\mathrm{L}\,, & x < 0\,, \\ u_\mathrm{R}\,, & x > 0\,, \end{cases}$$

where ρ_L, ρ_R, p_L, p_R, u_L and u_R are constants.

(i) Explain why the subsequent flow is of the similarity form described in part (a).

(ii) What constraints must be satisfied by the initial values if the subsequent evolution is to involve only two uniform states connected by a rarefaction wave? Give a non-trivial example of such a solution.

(iii) Explain why, for more general choices of the initial values, the solution cannot have the simple form described in part (ii), even if $u_\mathrm{R} > u_\mathrm{L}$. What other features will appear in the solution? (Detailed calculations are not required.)

4 The linearized equation governing the displacement $\boldsymbol{\xi}$ of a non-self-gravitating fluid with respect to a magnetostatic equilibrium state may be written

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathcal{F}\boldsymbol{\xi},$$

where $\mathcal{F}$ is a linear differential operator defined by

$$(\mathcal{F}\boldsymbol{\xi})_i = -\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\xi_j + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(V_{ijkl}\frac{\partial \xi_k}{\partial x_l}\right),$$

with

$$V_{ijkl} = \left(\gamma p + \frac{B^2}{\mu_0}\right)\delta_{ij}\delta_{kl} + \left(p + \frac{B^2}{2\mu_0}\right)(\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}) + \frac{1}{\mu_0}(B_j B_l \delta_{ik} - B_i B_j \delta_{kl} - B_k B_l \delta_{ij}).$$

(a) Show that $\mathcal{F}$ is self-adjoint in the sense that

$$\langle \boldsymbol{\eta}, \mathcal{F}\boldsymbol{\xi}\rangle = \langle \mathcal{F}\boldsymbol{\eta}, \boldsymbol{\xi}\rangle,$$

where

$$\langle \boldsymbol{\eta}, \boldsymbol{\xi}\rangle = \int \rho \boldsymbol{\eta}^* \cdot \boldsymbol{\xi}\, \mathrm{d}V$$

defines an inner product between two complex displacement fields $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$, and the integration is over all space. You may assume that the fluid body is of finite size and is surrounded by a perfectly conducting fluid of negligible density and pressure, and that the magnetic field decays sufficiently rapidly at infinity.

(b) Show that the functional

$$W[\boldsymbol{\xi}] = -\frac{1}{2}\langle \boldsymbol{\xi}, \mathcal{F}\boldsymbol{\xi}\rangle$$

is real, and explain its role in the equation for the energy of the perturbation.

(c) Show that the magnetostatic equilibrium is stable if $W[\boldsymbol{\xi}]$ admits only positive values, while it is unstable if $W[\boldsymbol{\xi}]$ admits negative values. [*Hint*: Consider the second derivative with respect to time of $\ln\langle \boldsymbol{\xi}, \boldsymbol{\xi}\rangle$ for a perturbation of zero energy, if this exists. Alternatively, if you restrict attention to normal-mode solutions of the linearized equations, you should prove any variational principles that you use.]

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