UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2009 1:30 pm to 4:30 pm

PAPER 63

STRUCTURE AND EVOLUTION OF STARS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

You may use the equations and results given below without proof. *The symbols used in these equations have the meanings that were given in lecutres. Candidates are reminded of the equations of stellar structure in the form:*

$$\frac{dm}{dr} = 4\pi r^2 \rho \qquad \frac{dP}{dr} = -\frac{Gm\rho}{r^2} \qquad \frac{dL_{\rm r}}{dr} = 4\pi r^2 \rho \epsilon$$

In a radiative region

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_{\rm r}}{16\pi a c r^2 T^3} .$$

In a convective region

$$\frac{dT}{dr} = \frac{(\Gamma_2 - 1)\, T}{\Gamma_2 P} \frac{dP}{dr} ,$$

the luminosity, radius and effective temperature are related by

$$L = 4\pi R^2 \sigma T_{\rm e}^4 .$$

The equation of state for an ideal gas and radiation is

$$P = \frac{\mathcal{R}\rho T}{\mu} + \frac{aT^3}{3}$$

with $1/\mu = 2X + 3Y/4 + \mathcal{Z}/2$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A cluster of chemically homogeneous massive stars remain fully mixed but radiative as they evolve. The stellar material is an ideal gas and radiation pressure is negligible. The energy generation is due to hydrogen burning via the CNO cycle with $\epsilon = \epsilon_0 X \rho T^{13}$ and the opacity, $\kappa = \kappa_0(1 + X)$, depends only on the hydrogen mass fraction, X.

Show by homology that, for such stars of mass M, the radius $R \propto M^{\frac{3}{4}}$ and the luminosity $L \propto M^3$. Deduce the slope in the theoretical Hertzsprung-Russell diagram for this hydrogen burning zero-age main sequence and plot this.

If such stars are fully mixed for their entire hydrogen burning lifetime they eventually become helium stars. Show for a star of fixed mass that, as X decreases,

$$L \propto M^3(1 + X)^{-1}(3 + 5X)^{-4}$$

and find a similar relation for R.

By considering $\frac{d \log L}{d \log T_{\mathrm{e}}}$ determine how the star evolves away from its zero-age main sequence when X is initially 1. Indicate this on your HR diagram.

How does this compare with observations and what does this tell us about normal stellar evolution?

2 A red giant can be modelled by an isothermal degenerate helium core surrounded by a thin hydrogen burning shell, above which is a radiative region which is itself surrounded by a deep convective envelope. The core is of mass M_1 and radius R_1 related by

$$M_1^{1/3} R_1 = A = \text{const.}$$

At the base of the radiative region just above the core boundary is the thin hydrogen-burning shell which generates the entire luminosity L. The entire radiative envelope has a negligible mass while the convective envelope above it has a significant amount of mass. The opacity obeys $\kappa = \kappa_0 \rho^n / T^m$, with n and m constant. Show that, when radiation pressure is neglected, the relation between P and T in the radiative region is

$$P = C \left(T^{4+m+n} + T_0^{4+m+n} \right)^{1/(n+1)},$$

where

$$C = \left[\frac{16\pi ac G M_1}{3\kappa_0 L} \left(\frac{\mathcal{R}}{\mu} \right)^n \frac{(n+1)}{(n+m+4)} \right]^{1/(n+1)}$$

and T_0 is an appropriate constant of integration.

The temperature at the boundary between the radiative zone and the convective envelope is T_{b}. Show that

$$T_0 = T_{\mathrm{b}} \left\{ \left(\frac{\gamma - 1}{\gamma} \right) \left(\frac{4+m+n}{n+1} \right) - 1 \right\}^{1/(4+n+m)},$$

where γ is the ratio of specific heats throughout the convective zone.

Show that, in regions near the shell, well below the inner boundary of the convective envelope, where $T \gg T_{\mathrm{b}}$ and hence $T \gg T_0$, the dependence of temperature on radius r is approximately

$$T = \frac{\mu}{\mathcal{R}} \frac{G M_1 (n+1)}{(4+n+m) r}.$$

Use this to show that, when $n = 1$, $m = 3$ and the energy generation rate is given by $\epsilon = \epsilon_0 \rho T^{10}$, with $\epsilon_0 = \text{const}$,

$$L = \frac{4\pi}{13} C^2 \epsilon_0 \left(\frac{\mu}{\mathcal{R}} \right)^2 \left(\frac{\mu G M_1}{4\mathcal{R}} \right)^{16} \frac{1}{R_1^{13}}.$$

Hence show that the luminosity depends on the core mass as

$$L \propto M_1^{32/3}.$$

What happens to L if mass is removed from the stellar envelope?

3 a) At very high temperatures nuclear burning leads to an equilibrium distribution of isotopes around the iron group. In this nuclear statistical equilibrium, the abundance by number of an isotope ${}^{A}_{Z}S$ is,

$$N(A,Z) \propto n_{\mathrm{p}}^{Z} n_{\mathrm{n}}^{A-Z} T^{\frac{3}{2}(1-A)} \exp\left(\frac{Q(A,Z)}{kT}\right),$$

where $Q(A,Z)$ is the binding energy of the isotope, n_{p} and n_{n} are the number density of free protons and neutrons respectively.

Derive an expression for the ratio of the abundances of ^{56}Ni and ^{54}Fe via the reversible reaction,

$${}^{56}_{28}\mathrm{Ni} \rightleftharpoons {}^{54}_{26}\mathrm{Fe} + 2\mathrm{p}.$$

As the reactions proceed the ratio of the mean number of protons to the mean number of neutrons, $\frac{<Z>}{<N>}$, is preserved. When $\frac{<Z>}{<N>} = 1$ if all material is converted to either ^{56}Ni, ^{54}Fe or protons what is n_{p}? Estimate the temperature at which $\frac{N(^{56}\mathrm{Ni})}{(N(^{54}\mathrm{Fe}))^3}$ is a maximum.

Outline when this equation can be used to calculate the outcome of nuclear reactions in the life of a massive star. What are the factors that cause $\frac{<Z>}{<N>}$ to decrease and when are they important to consider?

b) In most nuclear reactions throughout a star's lifetime $\frac{<Z>}{<N>} = 1$. However during helium burning nitrogen reacts as follows,

$$\alpha + {}^{14}\mathrm{N} \rightarrow {}^{18}\mathrm{F} \rightarrow {}^{18}\mathrm{O} + \mathrm{e}^{+}$$
$${}^{18}\mathrm{O} + \alpha \rightarrow {}^{22}\mathrm{Ne},$$

which forms nuclei with a ratio of $\frac{<Z>}{<N>} = \frac{5}{6}$. Nitrogen becomes the most abundant element after hydrogen burning via the CNO cycle. Assume it contributes the entire metallicity of a star at that point. If ^{22}Ne is the only element that contributes excess neutrons in a carbon-oxygen white dwarf and everything other than ^{22}Ne is converted to ^{12}C and ^{16}O in equal amounts by mass, show that,

$$\frac{< Z >}{< N >} = \frac{11 - \mathcal{Z}}{11 + \mathcal{Z}},$$

where $\mathcal{Z}$ is the metallicity mass fraction.

A type Ia supernova progenitor is a carbon-oxygen white dwarf. Describe the light curve of a type Ia supernova and how it is powered.

If a zero metallicity supernova forms $1M_{\odot}$ of nickel, calculate to 2 significant figures how much nickel a solar metallicity ($\mathcal{Z} = \frac{1}{50}$) type Ia supernova produces assuming the equilibrium is reached at the same temperature and density and that the only products are ^{56}Ni and ^{54}Fe.

When the luminosity of a type Ia supernova is assumed to be a standard candle and calibrated at solar metallicity, show that the error in the distance to an extremely low metallicity type Ia supernova is about 3%. [*You may find it useful to know that* $(Q(56,28) - Q(54,26))/k = -2.7\times10^{10}$K and $\sqrt{550/523} = 1.03$]

4 a) A cataclysmic variable consists of a white dwarf of mass M_1 and a low-mass main-sequence companion of mass M_2 in a circular orbit with separation a. The main-sequence star is filling its Roche lobe and transferring mass to the white dwarf at a rate $\dot{M}_1 = -\dot{M}_2$. The mass ratio $q = \frac{M_2}{M_1} < 1$. The hydrostatic and thermal equilibrium radius of the main-sequence star can be approximated by

$$\frac{R_2}{R_\odot} = \frac{M_2}{M_\odot},$$

while for a suitable range of mass ratios the Roche-lobe radius R_L obeys

$$\frac{R_\mathrm{L}}{a} = 0.46\left(\frac{M_2}{M}\right)^{\frac{1}{3}},$$

where $M = M_1 + M_2$. Show that the period P of the binary is given by

$$\frac{P}{P_0} = \frac{M_2}{M_\odot}$$

for some constant P_0 that you need not evaluate.

The spin angular momentum of the stars can be neglected. Show that the orbital angular momentum is

$$J = \frac{M_1 M_2}{M} a^2 \Omega,$$

where $\Omega = 2\pi/P$ is the orbital angular velocity.

Find $\dot{R}_\mathrm{L}/R_\mathrm{L}$ as a function of $\dot{M}_2/M_2$ when $\dot{J} = 0$ and compare this with $\dot{R}_2/R_2$. Why would the mass transfer be dynamically unstable if $q > 4/3$.

Describe briefly one mechanism that can lead to angular momentum loss ($\dot{J} < 0$) and maintain mass transfer if $q < 4/3$.

b) In a classical nova once a layer of hydrogen-rich material of mass $\delta m \approx 10^{-4}\,M_\odot$ has accumulated on the surface of the white dwarf thermonuclear reactions ignite in the degenerate material. These expel the entire layer of mass δm from the system in a nova explosion over a time that is very short compared with the mass-transfer timescale. Assume that the orbit remains circular and show that the change in separation $\delta a/a = \delta m/M$ and that the change in Roche-lobe radius $\delta R_\mathrm{L}/R_\mathrm{L} = 4\delta m/3M$ to first order in $\delta m/M$.

Deduce that mass transfer ceases. Assuming that there is a constant rate of angular momentum loss $-\dot{J}$ until mass transfer is resumed and this $-\dot{J}$ remains the same until the next nova explosion show, again to first order in $\delta m/M$, that the ratio of the time spent detached t_d, during this interruption, to the time spent semi-detached t_s, while the hydrogen-rich layer is accumulating, is

$$\frac{t_\mathrm{d}}{t_\mathrm{s}} = \frac{2q}{(4-3q)(1+q)}.$$

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