UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2009 9:00 am to 12:00 pm

PAPER 62

ASTROPHYSICAL DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Show that the gravitational potential energy of a spherical system with density $\rho(r)$ is given by

$$W = -4\pi G \int_0^\infty r\rho(r)M(r)dr,$$

where $M(r)$ is the mass enclosed within radius r.

Show that the projected density of the system $\Sigma(R)$ as a function of projected radius R is

$$\Sigma(R) = 2\int_R^\infty \frac{\rho(r)rdr}{\sqrt{r^2 - R^2}}.$$

Show how to invert this using Abel transforms to obtain the density as

$$\rho(r) = -\frac{1}{\pi}\int_r^\infty \frac{d\Sigma}{dR}\frac{dR}{\sqrt{R^2 - r^2}}.$$

Let us define the strip density $S(x)$ such that $S(x)dx$ is the mass in a strip of width dx passing a radius x from the projected centre. Show that

$$S(x) = 2\int_x^\infty \frac{\Sigma(R)RdR}{\sqrt{R^2 - x^2}}.$$

Hence show that the strip density and the mass density are related by

$$\rho(x) = -\frac{1}{2\pi x}\frac{dS}{dx}.$$

Finally, show that the gravitational potential energy may be written

$$W = -2G\int_0^\infty [S(x)]^2 dx.$$

2 (a) Let f be the distribution function of a collisionless, stellar system with density ρ moving in a gravitational potential ϕ. Let us use angled brackets to define averages over the distribution function in velocity space, for example

$$\langle v_i \rangle = \frac{1}{\rho} \int v_i f \, d^3 v \, .$$

Derive the Jeans equations in the form

$$\rho \, \frac{\partial \langle v_j \rangle}{\partial t} + \rho \, \langle v_i \rangle \, \frac{\partial \langle v_j \rangle}{\partial x_i} = -\rho \, \frac{\partial \phi}{\partial x_j} - \frac{\partial (\rho \sigma_{ij}^2)}{\partial x_i} \, ,$$

where the velocity dispersion tensor $\sigma_{ij}^2 = \langle (v_i - \langle v_i \rangle)(v_j - \langle v_j \rangle) \rangle$. What is the analogue of the Jeans equations for a fluid system?

(b) Let us consider a tracer population of stars with density ρ moving in the gravitational potential of a spherical dark halo ϕ . The Jeans equations in spherical polar coordinates may be written

$$\frac{d\rho i \, \langle v_r^2 \rangle}{dr} + \frac{\rho}{r} \left(2 \, \langle v_r^2 \rangle - \langle v_\theta^2 \rangle - \langle v_\phi^2 \rangle \right) = -\rho \, \frac{d\phi}{dr}$$

It is known that the distribution function of the stars depends on binding energy E alone, and that the radial velocity dispersion is constant, $\langle v_r^2 \rangle = \sigma_0^2$

Show that

$$\rho = \rho_0 \exp \left(\frac{\psi}{\sigma_0^2} \right) ,$$

where ρ_0 is a constant and $\psi = -\phi$. Hence, demonstrate that the stellar population has a distribution function

$$f(E) = \frac{\rho_0}{(2 \, \pi \, \sigma_0^2)^{3/2}} \, \exp \left(\frac{E}{\sigma_0^2} \right)$$

What distribution of line-of-sight velocities is seen by a spectroscopist?

Hint: You may assume Eddington's formula in the form

$$f(E) = \frac{1}{\sqrt{8} \, \pi^2} \, \frac{d}{dE} \int_{-\infty}^{E} \frac{d\rho}{d\psi} \, \frac{d\psi}{\sqrt{E - \psi}}$$

and are reminded of the standard integral

$$\int_{-\infty}^{\infty} \exp \left(-x^2 \right) dx = \sqrt{\pi}$$

3 (a) Derive the virial theorem for a collisionless self-gravitating stellar system in the form

$$2T + W = 0,$$

where T is the total kinetic energy and W the total potential energy of the system.

Show that the heat capacity of the system is negative.

(b) Now consider a general system of point masses with position vectors $\mathbf{r}_i$, momentum vectors $\mathbf{p}_i$ and applied forces $\mathbf{F}_i$. Let G be the quantity

$$G = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i.$$

Show that

$$\frac{dG}{dt} = 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i$$

where T is the kinetic energy of the ensemble. By denoting time-averages as angled brackets, show that

$$2\langle T\rangle = \left\langle \sum_i \mathbf{F_i} \cdot \mathbf{r_i} \right\rangle$$

If the only forces are interparticle and derivable from a potential

$$W = \sum_{j\neq i} C|\mathbf{r}_i - \mathbf{r}_j|^{p+1},$$

where C and p are constants, then show that

$$2\langle T\rangle - (p+1)\langle W\rangle = 0$$

Hence, find the condition on p that the heat capacity of the system is negative.

Hint: You may assume without proof that

$$\lim_{\tau\to\infty} \frac{1}{\tau}\left[G(\tau) - G(0)\right] = 0$$

4 Consider the motion of a planet in the force field

$$F = -\frac{GM_\odot}{r^2} + \frac{C}{r^3}$$

where r is the radius, $G, M_\odot$ and C are constants. The planet may be considered to behave as a test particle.

Show that the equation of the orbit can be cast into the form

$$r = \frac{a(1-e^2)}{1 + e\cos\alpha\theta},$$

where the constants a, e and α should be defined. Describe the bound orbit when (i) $\alpha = 1$ and (ii) $\alpha \neq 1$. In each case, write down a set of the independent, isolating integrals of the motion.

When $\alpha \approx 1$, derive an approximate expression for the rate of precession of the perihelion in terms of

$$\eta = \frac{aC}{GM_\odot}$$

After the known perturbations of the other planets are taken into account, the perihelion of Mercury is observed to precess at the rate of $40''$ per century. Estimate the value of η that could account for this.

Mercury's eccentricity is 0.206 *and period is* 0.24 *yr*

END OF PAPER