UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2009 1:30 pm to 4:30 pm

PAPER 6

FINITE DIMENSIONAL LIE ALGEBRAS AND THEIR REPRESENTATIONS

*Attempt **ALL** questions.*

*There are **FIVE** questions in total.*

*Question **THREE** carries the most weight.*

All Lie algebras are over $\mathbb{C}$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Let $\mathfrak{g}$ be an *abelian* Lie algebra. Show that the irreducible representations of $\mathfrak{g}$ are one dimensional, and are parameterized by linear maps $\lambda : \mathfrak{g} \to \mathbb{C}$.

Must every indecomposable representation of $\mathfrak{g}$ be irreducible?

Now let V be an arbitrary finite dimensional representation of $\mathfrak{g}$. Describe a canonical decomposition $V = \oplus_\lambda V^\lambda$ into submodules.

(ii) Define what it means for a Lie algebra $\mathfrak{b}$ to be *solvable*.

Describe all *irreducible* representations of $\mathfrak{b}$. State clearly any theorems you use.

2

Let $\mathfrak{g}$ be a Lie algebra, equipped with a non-degenerate invariant bilinear form $(\, , \,)$. Define the Casimir element Ω, and show it commutes with $\mathfrak{g}$.

Now let $\mathfrak{g} = \mathfrak{sl}_2$. Define an invariant bilinear form, and compute the action of Ω on an irreducible n-dimensional representation.

3 Let

$$\mathfrak{g} = \mathfrak{sp}_{2n} = \left\{ A \in Mat_{2n} | AJ + JA^T = 0 \right\}$$

where $J = \begin{pmatrix} & & & & & 1 \\ & & & & \cdots & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdots & & & & \\ -1 & & & & & \end{pmatrix}$, and let $\mathfrak{t} =$ diagonal matrices in $\mathfrak{g}$.

(i) Decompose $\mathfrak{g}$ as a $\mathfrak{t}$-module, and hence write the roots R for $\mathfrak{g}$. Choose positive roots to be those occurring in upper triangular matrices. Write down the positive roots R^+, the simple roots π, the highest root θ, and the fundamental weights.

Write down ρ.

Draw the Dynkin diagram and label it by simple roots. Draw the extended Dynkin diagram.

(ii) Show that $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$.

(iii) For each root $\alpha \in R$, write the reflection $s_\alpha : \mathfrak{t} \to \mathfrak{t}$ explicitly. Describe the Weyl group W (you do not need to prove your answer).

(iv) Let $V = \mathbb{C}^{2n}$ be the standard representation of $\mathfrak{sp}_{2n}$. Draw the crystal of V, and of $V \otimes V$. Write the highest weight of each irreducible summand of $V \otimes V$.

4

(i) State the Weyl dimension formula, briefly defining the notation you use.

(ii) Draw the root system of G_2, and the fundamental weights ω_1, ω_2.

Write down the dimension of the irreducible representation with highest weight $n_1\omega_1 + n_2\omega_2, \quad n_1, n_2 \in \mathbb{N}$.

5

(i) Let V be a representation of $\mathfrak{sl}_2$. Define the character of V.

(ii) Now let $\mathfrak{g}$ be a semisimple Lie algebra. Define the *principal* $\mathfrak{sl}_2$ inside $\mathfrak{g}$, and hence the q-character of a representation of $\mathfrak{g}$.

(iii) Compute the q-character of the adjoint representation of the simple Lie algebra of type G_2.

END OF PAPER