UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   **Part III**

Monday, 8 June, 2009   1:30 pm to 3:30 pm

PAPER 59

INTRODUCTION TO TWISTOR THEORY

This is a two hour paper with ***FOUR*** *questions*

of which ***THREE*** *are to be attempted.*

Each question should take about 40 minutes.
Each question is marked out of 33 possible points.
There is one mark for a particularly good bit of proof
or comment available across al three questions
This gives a total of 100 possible marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

This question uses the conventions below.

$$[\nabla_a, \nabla_b] v_d = R_{abcd} v^c$$
$$R_{ab} = R_{acb}{}^c$$

The spacetime has dimension 4 over the reals and has signature $(+, -, -, -)$.

The connection is a metric connection.

(a) Write down the symmetries of the Riemann tensor and the Ricci tensor.

(b) Prove that $[\nabla_a, \nabla_b]$ may be written in the form

$$[\nabla_{AA'}, \nabla_{BB'}] = \epsilon_{A'B'} \triangle_{AB} + \epsilon_{AB} \triangle_{A'B'}$$

in two-component spinors, where $\triangle_{AB}$ and $\triangle_{A'B'}$ are symmetric in their indices. Write expressions for $\triangle_{AB}$ and $\triangle_{A'B'}$ in terms of $\nabla_{AB'}$.

(c) Prove that for any function f

$$\triangle_{AB} \left(f \ \alpha^C \beta^{C'} \right) = f \left(\beta^{C'} \ \triangle_{AB} \ \alpha^C + \alpha^C \ \triangle_{AB} \ \beta^{C'} \right) .$$

(d) By considering a simple anti-self-dual bivector

$$w_{ab} = \alpha_A \beta_B \epsilon_{A'B'}$$

and using the symmetries of the curvature prove

$$\triangle_{AB} \left(\alpha^{(A} \beta^{B)} \right) = 0 .$$

(e) Using the above information, or otherwise, explain why it is reasonable to write

$$\triangle_{AB} \alpha_C = X_{ABDC} \alpha^D$$

for some

$$X_{ABCD} = \Psi_{ABCD} - \Lambda \epsilon_{D(A} \epsilon_{B)C}$$

such that

$$X_{ABCD} = X_{(AB)(CD)}$$
$$\Psi_{ABCD} = \Psi_{(ABCD)}$$

where Λ is to be found in terms of contractions of X_{ABCD}.

2

Given that twistor space is $T \cong \mathbb{C}^4$ and $F_{12}(T)$ is the correspondence space between complexified compactified Minkowski space, M, and projective twistor space, use this correspondence,

$$\begin{array}{ccc} & F_{12} & \\ \nu \swarrow & & \searrow \mu \\ M & & PT \end{array}$$

and the twistor incidence equation,

$$\omega^A = ix^{AA'}\pi_{A'}$$

where

$$Z^\alpha = \left(\omega^A, \pi_{A'}\right)$$

is a twistor, to identify the following geometric relationships.

(a) Define the inverse correspondence of a point p in M via the above double fibration by

$$\hat{p} := \mu \circ \nu^{-1}(p)$$

Prove if p is a point in M then $\hat{p} \cong \mathbb{CP}^1$.

(b) Define the correspondence of a point q in PT via the above double fibration by

$$\tilde{q} := \nu \circ \mu^{-1}(q)$$

Prove if q is a point in PT then $\tilde{q} \cong \mathbb{CP}^2$.

(c) Define α- and β-planes and describe the intersection in M of

(i) 2 distinct α-planes.

(ii) an α-plane and a β-plane.

(d) If two null geodesics meet in real Minkowski space, what condition does this place on the corresponding twistors, X^α and Y^α in PN?

(e) Given two colinear null twistors X^α and Y^α in PN, what does this imply about the possible intersection of the corresponding alpha-planes in M?

(f) How is a null cone in real Minkowski space characterised in projective twistor space?

3

Consider the twistor space is $T = \mathbb{C}^4$ and the double fibration,

$$\begin{array}{ccc} & F_{12} & \\ \nu \swarrow & & \searrow \mu \\ F_2 & & F_1 \end{array}$$

(a) Let V be a complex vector space with $dimV = n$.

What is the dimension of a flag manifold $F_{d_1 \cdots d_m}(V)$?

Using your answer above find the dimensions of the spaces, F_1, F_2, F_{12} for the twistor space $T = \mathbb{C}^4$, and identify them physically.

(b) Given that $SL(4, \mathbb{C})$ acts transitively on $T - \{0\}$ prove $SL(4, \mathbb{C})$ acts transitively on flags $[L_1, \cdots, L_m]$.

(c) The action of $SL(4, \mathbb{C})$ is equivariant on the double fibration above.

Explain the meaning of this statement.

(d) Let $p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ be a point in F_1. Define $H_p := \{g \in SL(4, \mathbb{C}) : \ g(p) = p\}$.

Find the form of matrices in H_p and hence show

$$F_1 \cong SL(4, \mathbb{C})/H_p \, .$$

Does the quotient space form a group? Justify your answer.

4

Write an essay describing the Plucker embedding into $\mathbb{CP}^5$. Pay particular attention to the relationship between the Euclidean and Lorentzian real spacetimes and their conformal compactifications. Will the physicists' idea of Wick rotation sending $t \mapsto it$ be compatible with the conformal compactifications?

END OF PAPER