UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2009 1:30 pm to 4:30 pm

PAPER 58

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A surface Σ, given by $f(x^a) = 0$, is constructed in a space $\mathcal{M}$. In $\mathcal{M}$, the metric tensor is g_{ab}, not necessarily positive definite, and its curvature tensor is R_{abcd}. Derive the Gauss-Codazzi equation for the curvature tensor formed from the metric induced on the surface Σ, carefully describing the geometric meaning of any quantities you use.

Consider five-dimensional Minkowski spacetime with metric

$$ds^2 = -dt^2 + dw^2 + dx^2 + dy^2 + dz^2.$$

The surface Σ, specified by the equation $-t^2 + w^2 + x^2 + y^2 + z^2 = 1$, is embedded in Minkowski spacetime. Find the curvature tensor of Σ in terms of h_{ab}, the metric induced on Σ. What is the sign of the Ricci scalar?

How could you modify this construction to find the opposite sign for the curvature on Σ?

2 Describe how to construct the covariant derivative of a spinor field ϵ.

In three spacetime dimensions, spinors are two dimensional and the gamma matrices can be taken to be

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Consider the spacetime metric

$$ds^2 = -V(r)^2 dt^2 + dr^2/V(r)^2 + r^2 d\theta^2$$

where t is a time coordinate, r a radial coordinate and θ an angular coordinate. Show that there are no solutions of

$$\nabla_a \epsilon = 0,$$

unless $V(r)$ is a constant.

If $V(r)$ is a constant, what conditions need to be put on θ to ensure that $r = 0$ is not a singularity? Carefully justify your answer.

Find all solutions to $\nabla_a \epsilon = 0$ in the case $V(r)$ is a constant.

3 The five-dimensional Chern-Simons action for a gauge field A, a one-form with values in the adjoint representation of a Lie group G, is

$$I_{CS} = \int \mathrm{Tr}\Big(F \wedge F \wedge A - \frac{1}{2} F \wedge A \wedge A \wedge A + \frac{1}{10} A \wedge A \wedge A \wedge A \wedge A \Big),$$

where F is the field strength for A.

Calculate the variation of the action, δI_{CS} under an infinitesimal variation, δA, of A.

What are the equations of motion for this theory?

Suppose one makes a gauge transformation on a solution of this equation of motion, does it remain a solution of the equation of motion? Give a clear justification of your answer.

4 A conformally flat metric can be written with the line element

$$ds^2 = \Omega^2 \; \delta_{ij} dx^i dx^j$$

where δ_{ij} is the metric on flat $\mathbb{R}^4$ and Ω is an arbitrary function of the coordinates x^i. Using the orthonormal basis

$$E^i = \Omega dx^i$$

show that the connection 1-form is given by

$$\omega^i{}_j = \Omega^{-2} \Big(E^i \Omega_j - E^k \Omega_l \, \delta_{jk} \, \delta^{il} \Big)$$

where

$$\Omega_i = \frac{\partial \Omega}{\partial x^i},$$

carefully explaining your reasoning.

Find an expression for the curvature tensor, the Ricci tensor and the Ricci scalar.

A four-dimensional metric of interest in quantum gravity is the Tolman wormhole which has line element

$$ds^2 = \left(1 + \frac{\lambda}{r^2} \right)^2 (dw^2 + dx^2 + dy^2 + dz^2)$$

where λ is a constant, w, x, y and z are coordinates on $\mathbb{R}^4$, and

$$r = \sqrt{w^2 + x^2 + y^2 + z^2}.$$

Does this solution satisfy the vacuum Einstein equations?

5 In a version of Kaluza-Klein theory, one can write the five-dimensional metric as

$$ds_5^2 = \gamma_{ij} dx^i dx^j + (dz + A_i dx^i)^2$$

where z is a periodic coordinate of period $2\pi R$, A_i is the "electromagnetic" field and γ_{ij} is the four-dimensional metric of the spacetime we directly see. A_i and γ_{ij} are independent of z.

Consider a particle moving along a five-dimensional geodesic. Derive its equation of motion in terms of the four-dimensional quantities A_i and γ_{ij} and objects constructed from them. Describe the physical meaning of the various terms in this equation, in particular describe carefully how the Lorentz force law appears in your equation.

By considering the Klein-Gordon equation in five dimensions, explain how the quantization of electric charge comes about and evaluate the fundamental unit of charge.

END OF PAPER