UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2009 9:00 am to 11:00 am

PAPER 57

ADVANCED COSMOLOGY

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In the 3+1 formalism for general relativity, we can define the extrinsic curvature of constant-time hypersurfaces Σ as

$$K_{\alpha\beta} = P^{\mu}{}_{\alpha} P^{\nu}{}_{\beta} \nabla_{\mu} n_{\nu} \,,$$

where the projector $P^{\mu}{}_{\alpha} = \delta^{\mu}{}_{\alpha} + n^{\mu} n_{\alpha}$ and n^{μ} is the future-pointing unit-normal to Σ which obeys $g_{\mu\nu} n^{\mu} n^{\nu} = -1$.

(i) Calculate n^{μ} and $K_{\alpha\beta}$ explicitly for a flat FRW universe with line element

$$ds^2 = a^2(\tau) \left[-d\tau^2 + d\mathbf{x}^2 \right] \,.$$

Briefly discuss the significance for this model of the trace $K = g^{\alpha\beta} K_{\alpha\beta}$. [Here, you may assume $\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (g_{\mu\lambda,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})$.]

(ii) The derivative operator $\mathcal{D}_{\alpha}$ on the hypersurface Σ can be defined from the $4D$ derivative ∇_{ν} by projecting all indices onto Σ using the projector $P^{\mu}{}_{\alpha}$. Show that $\mathcal{D}_{\alpha}$ satisfies

$$\mathcal{D}_{\gamma} \left({}^{(3)}g_{\alpha\beta} \right) = 0$$

where ${}^{(3)}g_{\alpha\beta} = g_{\alpha\beta} + n_{\alpha} n_{\beta}$ is the induced three-metric on Σ.

In addition, establish the following three simple identities:

$$n^{\mu} \nabla_{\nu} n_{\mu} = 0 \,, \qquad K_{\alpha\beta} = P^{\mu}{}_{\alpha} \nabla_{\mu} n_{\beta} \,,$$
$$P^{\mu}{}_{\alpha} P^{\nu}{}_{\beta} \nabla_{\mu} P^{\lambda}{}_{\nu} = K_{\alpha\beta}\, n^{\lambda}.$$

(iii) Using the results of (ii), or otherwise, derive the Codazzi equation

$$\mathcal{D}_{\beta} K^{\beta}{}_{\alpha} - \mathcal{D}_{\alpha} K = P^{\mu}{}_{\alpha} R_{\mu\nu} n^{\nu} \,,$$

which relates the extrinsic curvature $K_{\alpha\beta}$ on Σ to the $4D$ Ricci curvature $R_{\mu\nu}$. [Here, you may assume that $\nabla_{\nu} \nabla_{\mu} v^{\nu} - \nabla_{\mu} \nabla_{\nu} v^{\nu} = R_{\mu\lambda} v^{\lambda}$ for any vector v^{μ}.]

Briefly discuss the physical significance of the Codazzi equation as it relates to Einstein's equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$ for the flat FRW model described in (i). Suggest variables describing the *scalar* degrees of freedom which would be suitable for linearising the Codazzi equation about the flat FRW model.

2 In synchronous gauge for linear perturbations about a flat ($k = 0$) FRW background, the metric is taken to be $ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j\right]$, where $|\det(h_{ij})| \ll 1$ and primes (e.g. a') denote differentiation with respect to the conformal time τ.

(i) The coordinate transformation $(t,\ x^i) \longrightarrow (\tilde{t},\ \tilde{x}^i) = (t + \xi^0,\ x^i + \xi^i)$, (where the scalar part is $\xi^i \equiv \partial^i \lambda$), induces a gauge transformation in the metric perturbations,

$$\delta\tilde{g}_{\mu\nu} \;=\; \delta g_{\mu\nu} - \bar{g}_{\mu\nu,\lambda}\, \xi^\lambda - \bar{g}_{\lambda\nu}\, \xi^\lambda_{,\mu} - \bar{g}_{\mu\lambda}\, \xi^\lambda_{,\nu}\,.$$

Show that there is a residual gauge freedom in synchronous gauge given by the coordinate transformation,

$$\xi^0 \;=\; \frac{C(x^i)}{\bar{a}}\,, \qquad\qquad \lambda \;=\; C(x^i) \int \frac{1}{a}\, d\tau \;+\; D(x^i)\,,$$

where C and D are arbitrary functions of x^i only. Briefly discuss the significance of this gauge freedom for the density perturbation $\delta \equiv \delta\rho/\bar{\rho}$ during the standard hot big bang.

(ii) In a comoving frame, the cold dark matter density perturbation $\delta_{\rm c}$ and the trace of the metric perturbation $h = h_{ii}$ are related by $\delta_{\rm c} = -\frac{1}{2}h$ and the density perturbation evolution equation becomes

$$\delta_{\rm c}'' \;+\; \frac{a'}{a}\, \delta_{\rm c}' \;-\; \frac{3}{2}\left(\frac{a'}{a}\right)^2 \Omega_{\rm c}\, \delta_{\rm c} \;=\; 0\,, \qquad (\dagger)$$

where the density parameter is $\Omega_{\rm c} \equiv (8\pi\, G\, \bar{\rho}_{\rm c}/3)(a/a')^2$.

- Show that the density perturbation in synchronous gauge here is directly related to changes in the comoving volume.
- Find the growing and decaying mode solutions for $\delta_{\rm c}$ in the matter-dominated era, that is, after equal-matter radiation when $a \propto \tau^2$ ($\tau > \tau_{\rm eq}$).
- During the radiation era ($\tau < \tau_{\rm eq}$), adiabatic cold dark matter perturbations on superhorizon scales grow as $\delta_{\rm c} \propto \tau^2$ ($k\tau < 2\pi$), but subhorizon perturbations stagnate, $\delta_{\rm c} \approx$ const. ($k\tau > 2\pi$). Provide brief physical explanations for these different behaviours.

(iii) In synchronous gauge, the primordial adiabatic perturbation is $\zeta \equiv \frac{1}{6}\, h^- + \frac{1}{3}\, \delta_{\rm c}$. Given that the metric (curvature) perturbation h^- satisfies

$$\frac{a'}{a}\, h' + \frac{1}{3}\, k^2\, h^- \;=\; 3\left(\frac{a'}{a}\right)^2 \delta_{\rm c}\,,$$

show, using the growing mode solution found in (ii), that ζ is approximately constant on superhorizon scales during the matter era ($k\tau \ll 1$). Hence, or otherwise, derive a transfer function $T(k)$ in Fourier space to project forward a primordial adiabatic perturbation to give the density perturbation today, that is,

$$\delta_{\rm c}(\mathbf{k}, \tau_0) \;=\; T(k)\, \zeta(\mathbf{k}, \tau_i)\,,$$

in the case where the perturbation comes inside the horizon after matter-radiation equality ($\tau > \tau_{\rm eq}$). Find and justify the analogous transfer function for perturbations coming inside the horizon prior to equal matter-radiation ($\tau < \tau$eq). For a primordial scale-invariant power spectrum $P_\zeta(k, \tau_i) \equiv \langle|\zeta(\mathbf{k}, \tau_i)|^2\rangle = A/k^3$ (A is a constant), use this transfer function to find the power spectrum for $\delta_{\rm c}$ today.

3 The collisional Boltzmann equation (assuming isotropic scattering) for the photon brightness Δ in synchronous gauge is

$$\Delta' + ik\mu\Delta = -\tfrac{4}{3}\left[\tfrac{1}{2}h' + \tfrac{1}{2}(3\mu^2-1)h'_{\rm S}\right] + a\sigma_{\rm T}n_{\rm e}(\delta_\gamma + 4\mu(\theta_\gamma - \theta_{\rm b})/k^2 - \Delta)\,, \qquad (*)$$

where $\mathbf{k}$ is the wavevector ($k = |\mathbf{k}|$, $\hat{\mathbf{k}} = \mathbf{k}/k$)), $\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{n}}$ with $\hat{\mathbf{n}}$ the photon propagation direction, σ_T is the Thomson cross-section, $n_{\rm e}$ is the electron density, h and $h_{\rm S}$ are the scalar metric perturbations, δ_γ is the photon density perturbation, and θ_γ and $\theta_{\rm b}$ are the velocity potentials for the photons and baryons respectively. Ignoring metric perturbations, moment expansion of $(*)$ yields equations for δ_γ, θ_γ and the shear viscosity σ_γ (terminating at $\ell = 2$):

$$\delta'_\gamma - \tfrac{4}{3}k^2\theta_\gamma = 0\,, \qquad \theta'_\gamma + \tfrac{1}{4}\delta_\gamma - \sigma_\gamma = -an_{\rm e}\sigma_{\rm T}(\theta_\gamma - \theta_{\rm b})/k^2\,,$$
$$\sigma'_\gamma + \tfrac{4}{15}k^2\theta_\gamma = -an_{\rm e}\sigma_{\rm T}\sigma_\gamma\,.$$

The corresponding equations for the coupled baryons are

$$\delta'_{\rm b} + k^2\theta_{\rm b} = 0\,, \qquad \theta'_{\rm b} + \tfrac{a'}{a}\theta_{\rm b} + c_{\rm sb}^2\delta_{\rm b} = -Ran_{\rm e}\sigma_{\rm T}(\theta_{\rm b} - \theta_\gamma)/k^2\,,$$

where $c_{\rm sb}$ is the baryon sound speed and R is given in terms of the relative background photon and baryon densities $R = (4/3)\bar\rho_\gamma/\bar\rho_{\rm b}$.

(i) Briefly explain the origin of the collision terms and the metric perturbation terms on the right hand side of equation $(*)$. [Do not derive these.] Why can we ignore higher moments ($\ell \geqslant 2$) if the collision term $a\sigma_{\rm T}n_{\rm e}$ is large?

(ii) For initially adiabatic perturbations, while the photons and baryons are tightly coupled we will have $\delta_\gamma \approx \frac{4}{3}\delta_{\rm b}$ and $\theta_\gamma \approx \theta_{\rm b}$. Show that in this limit, the photon and baryon evolution equations can be combined to become

$$\delta'_\gamma = \frac{4}{3}k^2\theta_\gamma\,, \qquad \theta'_\gamma = -3c_{\rm s}^2\left(\tfrac{1}{4}\delta_\gamma - \sigma_\gamma\right) - \frac{3c_{\rm s}^2}{R}\left(\frac{a'}{a}\theta_\gamma + \frac{3}{4}c_{\rm sb}^2\delta_\gamma\right)\,,$$

where the effective sound speed is given by $c_{\rm s}^2 \equiv \frac{1}{3}\frac{R}{1+R} = \frac{1}{3}\left(1 + \frac{3}{4}\frac{\bar\rho_{\rm b}}{\bar\rho_\gamma}\right)^{-1}$.

(iii) In the limit that decoupling is in the matter era (cold dark matter $\Omega_{\rm c} \approx 1$ and $a \propto \tau^2$) with $\bar\rho_\gamma \gg \bar\rho_{\rm b}$, show that the tight coupling equations can be combined to yield

$$\delta''_\gamma - \tfrac{4}{3}k^2\sigma_\gamma + \tfrac{1}{3}k^2\delta_\gamma = 0\,.$$

Assuming that $\sigma'_\gamma \approx 0$, show that this becomes

$$\delta''_\gamma + \frac{4}{15}\tau_{\rm c}k^2\delta'_\gamma + \frac{1}{3}k^2\delta_\gamma = 0\,,$$

where $\tau_{\rm c} = (a\sigma_{\rm T}n_{\rm e})^{-1}$. Find general solutions of this equation (ignoring terms of $\mathcal{O}(\tau_{\rm c}^2)$) and show that for subhorizon scales they take the following approximate form

$$\delta_\gamma(\mathbf{k},\tau) \approx \left[A(\mathbf{k})\cos(k\tau/\sqrt{3}) + B(\mathbf{k})\sin(k\tau/\sqrt{3}))\right]\,\exp(-k^2/k_{\rm D}^2)\,,$$

where $k_{\rm D}$ is a time-dependent damping scale which you should specify.

(iv) Use the solution obtained in (iii), or otherwise, to describe the effect on the angular power spectrum of the temperature anisotropy $\frac{\Delta T}{T}$ which arises from intrinisic photon density fluctuations and relative motions.

END OF PAPER