UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2009 1:30 pm to 4:30 pm

PAPER 56

BLACK HOLES

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A variant of general relativity admits an electrically charged black hole solution with metric

$$ds^2 = -H(r)^{-1/2}F(r)dt^2 + H(r)^{1/2}\left[F(r)^{-1}dr^2 + r^2 d\Omega^2\right],$$

$$H(r) = 1 + \frac{2m\sinh^2\alpha}{r}, \qquad F(r) = 1 - \frac{2m}{r},$$

where m and α are real constants with $m > 0$.

(a) Calculate the Komar mass of this solution. **[4]**

(b) Show that one can define a quantity r_* such that $u = t - r_*$ and $v = t + r_*$ are constant on outgoing and ingoing radial null geodesics respectively. (You may express r_* as an integral.) **[2]**

(c) Obtain the above metric in ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ). Hence show that it can be analytically extended through $r = 2m$. **[2]**

(d) Define the *black hole region* of an asymptotically flat spacetime. Prove that the region $r < 2m$ (in ingoing Eddington-Finkelstein coordinates) is within the black hole region, and that the region $r > 2m$ does not intersect the black hole region. **[5]**

(e) Show that the surface $r = 2m$ is a Killing horizon of the stationary Killing vector field, and determine the surface gravity κ. **[5]**

(f) Define Kruskal coordinates $U = -e^{-\kappa u}$, $V = e^{\kappa v}$. Hence show that $r = 2m$ is a bifurcate Killing horizon. (You will need to consider the behaviour of r^* for $r \approx 2m$.) **[8]**

(g) Deduce the Penrose diagram for this solution. (You may assume that there is a curvature singularity at $r = 0$.) Describe briefly how the global structure of this solution differs from that of an non-extreme electrically charged Reissner-Nordstrom black hole. **[4]**

2

(a) Define what it means for an asymptotically flat spacetime to be *stationary and axisymmetric.* **[2]**

(b) Consider an isolated uncharged star that undergoes gravitational collapse to form a black hole. Explain carefully why the final state is characterized by only two parameters. **[6]**

(c) In Boyer-Lindquist coordinates, the Kerr solution is

$$\begin{aligned} ds^2 &= -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} dt^2 - 2a\sin^2\theta \left(\frac{r^2+a^2-\Delta}{\Sigma}\right) dt\, d\phi \\ &+ \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \end{aligned}$$

where $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2\cos^2\theta$.

Consider the case $M > a > 0$ describing a Kerr black hole.

(i) Show that a timelike or null geodesic with zero energy cannot intersect the region outside the ergosphere. **[2]**

(ii) Is the outer surface of the ergosphere a Killing horizon? Explain your answer. **[3]**

(d) For the Kerr solution with $0 < M < a$, Kerr-Schild coordinates $(\tilde{t}, x, y, z)$ are defined by

$$\begin{aligned} x + iy &= (r + ia)\sin\theta \exp\left(i\phi + ia\int \frac{dr}{\Delta}\right), \\ z &= r\cos\theta \\ \tilde{t} &= t - r + \int \frac{(r^2+a^2)}{\Delta} dr. \end{aligned}$$

In these coordinates, the metric is

$$\begin{aligned} ds^2 &= -d\tilde{t}^2 + dx^2 + dy^2 + dz^2 \\ &+ \frac{2Mr^3}{r^4 + a^2 z^2}\left[\frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r} + d\tilde{t}\right]^2 \end{aligned}$$

(i) Describe how to construct a maximal analytic extension of this solution. You may assume that there is a curvature singularity where $\Sigma = 0$. **[10]**

(ii) Show that this spacetime contains closed timelike curves. **[4]**

(e) Could the closed timelike curves of the Kerr solution occur in Nature? Answer briefly, distinguishing between the cases $0 < M < a$ and $M > a > 0$. **[3]**

3

(a) What is a null geodesic congruence? Define the expansion θ, shear $\hat{\sigma}$ and rotation $\hat{\omega}$ of a null geodesic congruence. **[4]**

(b) Show that $\hat{\omega} = 0$ if, and only if, the congruence is hypersurface-orthogonal. **[3]**

(c) Explain the geometrical significance of θ and $\hat{\sigma}$ (you may restrict attention to the case $\hat{\omega} = 0$) and state an equation relating θ to the rate of increase of the area of an infinitesimal surface element. **[3]**

(d) In the Schwarzschild spacetime, consider the null geodesic congruence consisting of ingoing radial null geodesics. Let U^a be the tangent to the affinely parametrized geodesics.

(i) Find a vector field N^a obeying $N^2 = 0$, $N \cdot U = -1$ and $U \cdot \nabla N^a = 0$ everywhere. (Hint: use ingoing Eddington-Finkelstein coordinates, and assume that N^a is spherically symmetric, i.e. $N^\theta = N^\phi = 0$.) **[5]**

(ii) Calculate the expansion, shear and rotation of this congruence. **[5]**

(e) State the second law of black hole mechanics. Give a proof of the second law, starting from Raychaudhuri's equation:

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^2 + \hat{\omega}^2 - R_{ab}U^aU^b.$$

You should state clearly any result you assume regarding conjugate points. You may assume that the generators of the future horizon are complete to the future. **[10]**

4

(a) Write an essay giving a detailed account of the quantum theory of a free scalar field in a globally hyperbolic spacetime. You should explain carefully why the particle concept is ambiguous in general and why it can be made unambiguous in a stationary spactime. Describe how to calculate the expected number of particles produced in a spacetime that is stationary at early and late times but time-dependent in between. **[22]**

(b) Explain why the discovery that black holes emit thermal radiation implies that the laws of black hole mechanics can be reinterpreted in thermodynamical terms. **[8]**

END OF PAPER