UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2009 9:00 am to 12:00 pm

PAPER 55

COSMOLOGY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Robertson-Walker line element for a homogeneous, isotropic universe with vanishing spatial curvature is

$$ds^2 = dt^2 - a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \ .$$

(a) When the energy-momentum tensor takes a perfect fluid form with energy density ρ and pressure P, the Einstein equation implies that the scale factor $a(t)$ evolves as:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 \ ,$$

where the overdot denotes a derivative with respect to time t. Find the values of the redshift z_c such that galaxies observed in opposite directions (e.g. towards the North Pole and towards the South Pole) with $z > z_c$ cannot have communicated causally with each other in two cases:

1. when the energy density is dominated by non-relativistic matter with negligible pressure; and
2. when the energy density is dominated by radiation.

(b) Consider a universe filled with a fine network of domain walls (planar surfaces of constant energy per unit area) such that, coarse grained over macroscopic scales, the energy distribution is homogeneous and isotropic. Suppose that, as the universe expands, the surfaces of the domain walls and the network as a whole stretch uniformly in proportion to the scale factor, while the energy per unit area of the walls remains constant.

1. Suppose the energy density of the universe comes from these walls, and the universe is flat. By considering how the energy density varies with the scale factor, determine $H_0 t_0$ where H_0 is the present-day Hubble constant and t_0 is the current age of the universe.
2. Is this universe accelerating or decelerating?
3. At time t_0 (when the Hubble parameter is H_0) a light signal is sent out in the hope of getting back a return signal some time in the future. What is the maximum physical distance (as evaluated at time t_0) from which a return signal can be received, assuming that the return signal is sent out as soon as the incoming signal is received?

2 Recall that Einstein's general relativity is a theory of gravity described in vacuum by the Hilbert Lagrangian $L_H = -R_E/2$, where R_E is the Ricci scalar in Einstein gravity. Consider a modified theory of gravity, with a generalized Lagrangian

$$L_g = f(R) ,$$

where R is the Ricci scalar of this theory. In vacuum, the field equations obtained by varying the action associated with L_g with respect to the 4-dimensional spacetime metric $g_{\mu\nu}$ can be written in the form

$$f'\left(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}\right) + \tfrac{1}{2}g_{\mu\nu}\left(Rf' - f\right) - \nabla_\mu\nabla_\nu f' + g_{\mu\nu}\Box f' = 0 ,$$

where $\Box = g_{\mu\nu}\nabla^\mu\nabla^\nu$, ∇_μ is the covariant derivative, and $f' = df/dR$. Consider the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} ,$$

where Ω is a smooth, strictly positive function chosen such that $\Omega^2 = -f'(R)$. Under this transformation, the Ricci tensor transforms as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2}\frac{1}{f'^2}\nabla_\mu f'\nabla_\nu f' - \frac{1}{f'}\nabla_\mu\nabla_\nu f' - \frac{1}{2}g_{\mu\nu}\frac{1}{f'}\Box f' .$$

(a) Write down the transformed Ricci scalar $\tilde{R} = \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu}$. Hence, derive the transformed $\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}$.

(b) Let us introduce a scalar field ϕ, defined by $\phi = \sqrt{\frac{3}{2}}\ln\left[-f'(R)\right]$. By writing down the stress-energy tensor $T_{\mu\nu}$ for a scalar field with potential $V(\phi)$ minimally coupled to Einstein gravity with action S_ϕ,

$$T_{\mu\nu} = \frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S_\phi}{\delta\tilde{g}^{\mu\nu}} ,$$

or otherwise, show that the conformally-transformed vacuum field equations in $f(R)$ gravity correspond to the Einstein equations for a scalar field source with potential

$$V(\phi) = \frac{Rf' - f}{2f'^2} .$$

(c) What is the form of the potential corresponding to $f(R) = -(R + 2\Lambda)/2$, i.e. general relativity with a cosmological constant?

[*In this question, assume $\hbar = c = 8\pi G = 1$. Note also that the action for a minimally-coupled scalar field is*

$$S_\phi = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) .$$

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3 In the early universe the Hubble parameter is

$$H = \frac{1.66\sqrt{g^*}T^2}{m_{\mathrm{Pl}}} \, ,$$

where the Planck mass, $m_{\mathrm{Pl}} \approx 10^{19}\,\mathrm{GeV}$.

(a) A theory of weak interactions predicts a new, additional species of massless neutrino with interaction rate $\Gamma = G_A^2 T^5$, where $G_A = 10^{-12}\,\mathrm{GeV}^{-2}$. Estimate the temperature at which these neutrinos decouple from thermal equilibrium. (You may assume that $g^* \sim 100$ above the electroweak phase transition.)

(b) Evaluate the temperature today of these neutrinos relative to the photon temperature explaining your reasoning. (You will need to compute g^* at all appropriate epochs.) Estimate the proportion of the total entropy density in these new neutrinos.

(c) A variant of this theory predicts the new neutrinos to have a small mass. Discuss the constraints imposed by standard cosmology on the mass.

4 Consider a simple cosmological model described by linear, scalar perturbations about a spatially flat Robertson-Walker metric. The only matter present is an ideal fluid and the cosmological constant may be ignored.

(a) Explain why the perturbed line element can be written as

$$ds^2 = a^2(\eta)\left[(1+2\phi)d\eta^2 - (1-2\phi)\delta_{ij}dx^i dx^j\right] .$$

(b) The perturbed Einstein equations are

$$\begin{aligned}
\nabla^2\phi - 3\mathcal{H}\left(\dot{\phi} + \mathcal{H}\phi\right) &= 4\pi G a^2 \bar{\rho}\delta , \\
\dot{\phi} + \mathcal{H}\phi &= -4\pi G a^2(\bar{\rho} + \bar{P})v , \\
\ddot{\phi} + 3\mathcal{H}\dot{\phi} + \left(2\dot{\mathcal{H}} + \mathcal{H}^2\right)\phi &= 4\pi G a^2(\delta P) ,
\end{aligned}$$

where ρ is the energy density (with fractional perturbation δ), P is the pressure (with perturbation δP) and $\partial_i v$ is the peculiar velocity of the fluid. The unperturbed energy density and pressure are denoted by $\bar{\rho}$ and $\bar{P}$ respectively. Overdots denote differentiation with respect to conformal time η and $\mathcal{H} \equiv \dot{a}/a$.

For a fluid with constant equation of state w, i.e. $P = w\rho$, show that $a \propto \eta^{2/(1+3w)}$, for $w \neq -1/3$, and so $a^2\bar{\rho} \propto \eta^{-2}$. Hence, or otherwise, use the Einstein equations to show that the comoving-gauge density contrast, $\Delta \equiv \delta - 3\mathcal{H}(1 + \bar{P}/\bar{\rho})v$, evolves as

$$\ddot{\Delta} + \frac{2(1-3w)}{(1+3w)\eta}\dot{\Delta} - \frac{6(1-w)}{(1+3w)\eta^2}\Delta - w\nabla^2\Delta = 0 .$$

(c) For a radiation fluid ($w = 1/3$), solve this equation in Fourier space for a mode with comoving wavenumber k in the limiting cases that the mode is well outside the sound horizon and well inside. Assuming that Δ is regular at early times, describe the evolution of a given Fourier mode from early times ($k\eta/\sqrt{3} \ll 1$) to late times ($k\eta/\sqrt{3} \gg 1$).

(d) For a pressure-free fluid, show that Δ has solutions going like a and $a^{-3/2}$. Find how the peculiar velocity (potential) of the fluid, v, evolves with scale factor in the growing mode ($\Delta \propto a$).

5 The equations of motion for a flat Robertson-Walker cosmology containing a homogeneous scalar field, $\Phi(t)$, with potential energy density $V(\Phi)$ are

$$\partial_t^2\Phi + 3H\partial_t\Phi + V' = 0\ , \qquad H^2 = \frac{1}{3M_{\rm Pl}^2}\left(V + \frac{1}{2}(\partial_t\Phi)^2\right)\ ,$$

where $V' \equiv dV/d\Phi$, H is the Hubble parameter and $M_{\rm Pl}$ is the reduced Planck mass.

(a) Show that $2M_{\rm Pl}^2\partial_t H = -(\partial_t\Phi)^2$.

(b) Briefly describe the mechanism by which inflation produces fluctuations in the comoving curvature perturbation, $\mathcal{R}$, on super-Hubble scales.

(c) The power spectrum of these fluctuations is

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi\partial_t\Phi}\right)^2\ ,$$

where the right-hand side is evaluated at Hubble exit ($k = aH$). Using the slow-roll approximation, $(\partial_t\Phi)^2 \ll V$ and $3H\partial_t\Phi \approx -V'$, show that

$$\mathcal{P}_{\mathcal{R}}(k) \approx \frac{8}{3\epsilon_V}\left(\frac{V^{1/4}}{\sqrt{8\pi}M_{\rm Pl}}\right)^4\ ,$$

where $\epsilon_V \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2$.

(d) Show also that the spectral index, defined by $n_s(k) \equiv 1 + d\ln\mathcal{P}_{\mathcal{R}}(k)/d\ln k$, is

$$n_s(k) \approx 1 - 6\epsilon_V + 2\eta_V\ ,$$

where $\eta_V \equiv M_{\rm Pl}^2\frac{V''}{V}$.

(e) Show further that

$$dn_s/d\ln k \approx 16\epsilon_V\eta_V - 24\epsilon_V^2 - 2\xi_V^2\ ,$$

where $\xi_V^2 \equiv M_{\rm Pl}^4\frac{V'V'''}{V^2}$.

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