UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2009 1:30 pm to 4:30 pm

## PAPER 54

## GENERAL RELATIVITY

*Attempt no more than* ***THREE*** *questions.*

*There are* ***FOUR*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** State the strong principle of equivalence (SEP). Give experimental evidence for its validity. Explain how the SEP is incorporated into the theory of general relativity.

Show how Newtonian physics can be described in terms of a curved manifold and determine its curvature tensor.

Assume that spacetime contains a perfect fluid. By comparing with the Newtonian case, give a heuristic derivation of the Einstein equations.

**2** A vector field $\xi$ is Killing if and only if the spacetime metric $g_{ab}$ obeys $\mathcal{L}_\xi g_{ab} = 0$. Show that this implies Killing's equation

$$\xi_{(a;b)} = 0 ,$$

where the semicolon denotes the metric covariant derivative.

Starting from the Ricci identity

$$X^a{}_{;dc} - X^a{}_{;cd} = R^a{}_{bcd} X^b ,$$

show that for a Killing vector

$$\xi_{a;b}{}^{;b} + R_{ab}\, \xi^b = 0 .$$

Now suppose spacetime is axisymmetric, i.e. it has a Killing vector $\xi = \partial/\partial\phi$ in cylindrical polar coordinates $(t, r, z, \phi)$. Suppose also that spacetime consists of a compact self-gravitating matter source emitting gravitational waves. Within a spacelike hypersurface $t = \text{const}$, the angular momentum inside a closed 2-surface $S$ may be defined by

$$J_S = -\frac{1}{16\pi} \oint_S \xi^{a;b}\, \mathrm{d}^2\, \Sigma_{ab} , \qquad (\star)$$

where $\Sigma_{ab}$ is the proper area element on $S$. By considering two such 2-surfaces in the asymptotically flat region outside the matter source and using the Einstein equations, show that axisymmetric gravitational waves do not carry any angular momentum.

Define the total angular momentum by the integral $(\star)$ over a 2-surface $S$ at infinity. Show that this definition is independent of the spacelike hypersurface chosen.

[*You may use without proof the theorems of Stokes and Gauss in the following form: for an antisymmetric 2-tensor field* $F_{ab}$ *and vector field* $X^a$,

$$\oint_{\partial\Omega} F^{ab}\, \mathrm{d}^2\Sigma_{ab} = 2 \int_\Omega F^{ab}{}_{;b}\, \mathrm{d}^3\Sigma_a ,$$

$$\int_{\partial V} X^a\, \mathrm{d}^3\Sigma_a = \int_V X^a{}_{;a}\, \mathrm{d}^4\Sigma . ]$$

**3** This question is concerned with null geodesics in Schwarzschild spacetime with metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2).$$

Derive the equations of geodesic motion, clearly identifying any constants that you introduce. Show that without loss of generality, the geodesics may be assumed to lie in the equatorial plane $\theta = \pi/2$. Setting $u \equiv M/r = u(\phi)$, show that null geodesics obey

$$u'^2 = \alpha^2 - u^2 + 2u^3,$$

where $u' \equiv \mathrm{d}u/\mathrm{d}\phi$, and $\alpha$ is a constant that should be identified.

Show that a circular orbit is only possible at $r = 3M$.

Show also that there are orbits given by

$$\frac{1 - 3u}{(\sqrt{3} + \sqrt{1 + 6u})^2} = A\mathrm{e}^{\phi},$$

where $A$ is an arbitrary constant. Describe qualitatively the shape of these orbits as $\phi \to -\infty$, distinguishing between the different signs of $A$.

**4** A closed Friedmann-Robertson-Walker universe has metric

$$ds^2 = -\mathrm{d}t^2 + a^2(t)[\mathrm{d}\chi^2 + \sin^2\chi\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)].$$

You may assume without calculation that the Ricci tensor has nonzero components

$$R_{tt} = -3\frac{\ddot{a}}{a}, \quad R_{\chi\chi} = \frac{R_{\theta\theta}}{\sin^2\chi} = \frac{R_{\phi\phi}}{\sin^2\chi \sin^2\theta} = a\ddot{a} + 2\dot{a}^2 + 2.$$

Suppose the universe is a vacuum and satisfies the Einstein equations with a cosmological constant $\Lambda$. What sign must $\Lambda$ have? Obtain the scale factor $a(t)$. Choose the origin of $t$ such that $a(t)$ is time-symmetric and rescale $t$ by a constant to simplify the result. Identify this spacetime.

Find a conformally related spacetime by introducing a compactified time coordinate

$$T = 2\tan^{-1}(\mathrm{e}^t) - \tfrac{1}{2}\pi.$$

Draw the Penrose diagram, clearly identifying the boundaries and stating the range of the coordinates.

Explain the concepts of particle, event and Cauchy horizons and determine which of these exist in this spacetime.

**END OF PAPER**