UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 9 June, 2009 9:00 am to 11:00 am

PAPER 52

CONTROL OF QUANTUM SYSTEMS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Paradigms and Objectives for Quantum Control

(a) Sketch a simple block-diagram of a control system. Explain what is special (compared to the classical case) about the roles the sensors (measurements), actuators and environment play when the system to be controlled is governed by the laws of quantum physics. Briefly explain the difference between open-loop and closed-loop control.

(b) Describe three typical objectives of quantum control. (Give mathematical definitions.)

(c) Briefly explain in about one sentence each the general meaning of the notions of reachable sets and controllability in control theory, and explain how these notions can be applied to (Hamiltonian) quantum control systems, e.g., in the context of quantum state or process engineering.

(d) One paradigm for quantum control is Hamiltonian engineering. Briefly explain the general idea as well as three different approaches to (model-based) open-loop Hamiltonian engineering discussed in the lectures.

(e) Open-loop Hamiltonian engineering entirely based on a model of the system struggles in the presence of model uncertainty. Explain how we can avoid or mitigate this problem using adaptive closed-loop experiments. Consider how to formulate the problem mathematically as an optimal control problem and how we might solve the latter without recourse to a model, using only experimental data.

2 Bloch vectors and Bloch equation.

Let $\mathcal{H}$ be an N-dimensional (complex) Hilbert space and $\mathfrak{B}(\mathcal{H})$ be bounded operators (matrices) on $\mathcal{H}$. Define the Hilbert-Schmidt (HS) inner product for operators $A, B \in \mathfrak{B}(\mathcal{H})$ by $\langle A|B\rangle = \mathrm{Tr}(A^\dagger B)$ and let $\{\sigma_k\}_{k=1}^{N^2}$ be an orthonormal basis (w.r.t. the HS inner product) for the Hermitian matrices on $\mathcal{H}$, with $\sigma_{N^2} = \frac{1}{\sqrt{N}}\mathbb{I}$, where $\mathbb{I}$ is the identity matrix.

(a) Show that any operator $A \in \mathfrak{B}(\mathcal{H})$ can be written as $A = \sum_{k=1}^{N^2} a_k \sigma_k$ with $a_k = \mathrm{Tr}(\sigma_k A)$, and show that the coordinate vector $\mathbf{a} = (a_1, \ldots, a_{N^2})$ is real if A is Hermitian.

(b) Show that the coordinate vector for any density operator ρ is real with $r_{N^2} = \frac{1}{\sqrt{N}}$, and that the Bloch vector, $\mathbf{s} = (r_1, \ldots, r_{N^2-1})$ satisfies $\|\mathbf{s}\| \leqslant \sqrt{1-1/N}$, where $\|A\| = \sqrt{\mathrm{Tr}(A^\dagger A)}$ is the HS norm, with equality if and only if ρ is a rank-1 projector (pure state).

(c) The previous part shows that all pure states are contained in a sphere of radius $\sqrt{1-1/N}$ in $\mathbb{R}^{N^2-1}$. Show that the set of pure states is a proper subset of the sphere except for $N = 2$.

(d) Let us consider the dynamics. Assume ρ satisfies the quantum Liouville equation ($\hbar = 1$)

$$\dot\rho(t) = -i[H, \rho(t)] + \sum_d \mathcal{D}[V_d]\rho(t), \quad \mathcal{D}[V]\rho = V\rho V^\dagger - \frac{1}{2}(V^\dagger V\rho + \rho V^\dagger V). \tag{1}$$

Show that the coordinate vector $\mathbf{r}$ of ρ satisfies the matrix differential equation $\dot{\mathbf{r}}(t) = (\mathbf{L} + \sum_d \mathbf{D}^{(d)})\mathbf{r}(t)$ where $\mathbf{L}$ and $\mathbf{D}^{(d)}$ are $N^2 \times N^2$ matrices with

$$L_{mn} = \mathrm{Tr}(+iH[\sigma_m, \sigma_n]), \quad D_{mn}^{(d)} = \mathrm{Tr}(V_d^\dagger \sigma_m V_d \sigma_n) - \frac{1}{2}\mathrm{Tr}(V_d^\dagger V_d\{\sigma_m, \sigma_n\}) \tag{2}$$

where $[A, B] = AB - BA$ is the usual matrix commutator and $\{A, B\} = AB + BA$ the anticommutator. Furthermore show that $\dot r_{N^2} = 0$ and Bloch vector $\mathbf{s}$ satisfies an affine linear equation $\dot{\mathbf{s}} = \mathbf{A}\,\mathbf{s} + \mathbf{c}$.

(e) Show that for Hamiltonian systems, i.e., all $V_d = 0$, the evolution of the Bloch vector corresponds to a rotation. (Hint: Show that $\mathbf{L}$ and hence $\mathbf{A}$ is real anti-symmetric and $\mathbf{c} = 0$.)

(f) Assume $V_d = 0$ for all d (Hamiltonian dynamics) and the Hamiltonian $H = H[\mathbf{f}]$ depends on external controls $\mathbf{f}$ in such a way that the system is (density matrix) controllable. Is the state $\mathbf{s}_1 = (0, 0, 1)/\sqrt{2}$ reachable from the initial state $\mathbf{s}_0 = (0.5, 0.5, 0)/\sqrt{2}$ for $N = 2$? Explain why or why not.

3 Markovian reservoir engineering.

Let ρ be a density operator on a (complex) Hilbert space $\mathcal{H}$ of dimension $N < \infty$, and let $\mathbf{s} = (s_1, \ldots, s_{N^2-1})$ with $s_k = \mathrm{Tr}(\rho\sigma_k)$ be the coordinate vector (Bloch vector) with respect to a (not necessarily orthonormal) basis $\{\sigma_k\}_{k=1}^{N^2-1}$ for the trace-zero Hermitian matrices on $\mathcal{H}$. If ρ satisfies the quantum Liouville equation ($\hbar = 1$)

$$\dot{\rho}(t) = -i[H, \rho(t)] + \sum_d \mathcal{D}[V_d]\rho(t), \quad \mathcal{D}[V]\rho = V\rho V^\dagger - \frac{1}{2}\{V^\dagger V, \rho\}, \tag{1}$$

where $[A, B] = AB - BA$ is the usual matrix commutator and $\{A, B\} = AB + BA$ the anticommutator, then the corresponding Bloch vector $\mathbf{s}$ satisfies an affine-linear matrix differential equation $\dot{s}(t) = \mathbf{A}\,\mathbf{s} + \mathbf{c}$. (You may use this result without proof.)

(a) Define the notion of a steady state and characterize the steady states of the Bloch equation $\dot{\mathbf{s}}(t) = \mathbf{A}\,\mathbf{s}(t) + \mathbf{c}$ as a linear equation in terms of, e.g., the rank of $\mathbf{A}$. Does the Bloch equation always have a steady state? When does the Bloch equation have a unique steady state?

(b) Give a necessary and sufficient condition for attractivity of a steady states in terms of the eigenvalues of $\mathbf{A}$.

(c) The master equation for a system subject to measurement of the operator M and direct feedback with feedback Hamiltonian F is given by

$$\dot{\rho}(t) = -i[H_{\mathrm{tot}}, \rho(t)] + \mathcal{D}[M - iF]\rho(t) \tag{2}$$

where $H_{\mathrm{tot}} = H_0 + H_c + \frac{1}{2}(M^\dagger F + FM)$. Using this result and assuming $H_0 = 0$, $H_c = \alpha\sigma_y$, $M = \sigma$ and $F = \lambda\sigma_y$, where

$$\sigma = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_y = i\sigma - i\sigma^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{3}$$

show that the direct feedback master equation is $\dot{\rho}(t) = -i[\alpha\sigma_y, \rho(t)] + \mathcal{D}[\sigma - i\lambda\sigma_y]\rho(t)$.

(d) Expanding the single qubit feedback master equation above with respect to the basis $\sigma_x = \sigma + \sigma^\dagger$, σ_y, $\sigma_z = \frac{1}{2}[\sigma_x, \sigma_y]$, we obtain the Bloch equation

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} (2\lambda+1)^2 & 0 & -4\alpha \\ 0 & 1 & 0 \\ 4\alpha & 0 & (2\lambda+1)^2+1 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2\lambda+1 \end{pmatrix}, \tag{4}$$

where $\star(t) = \mathrm{Tr}(\sigma_\star\rho(t))$ for $\star \in \{x, y, z\}$ as usual. As $y(t)$ is decoupled (independent of x and z) and decays exponentially ($y(t) = y_0 e^{-1/2t}$) regardless of the feedback and control Hamiltonian, consider the reduced dynamics in the (x, z) plane

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} (2\lambda+1)^2 & -4\alpha \\ 4\alpha & (2\lambda+1)^2+1 \end{pmatrix}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} - \begin{pmatrix} 0 \\ 2\lambda+1 \end{pmatrix}. \tag{5}$$

Show that except for $(\alpha, \lambda) = (0, -\frac{1}{2})$, the reduced system has a unique steady state given by

$$x_{ss} = -8\alpha(2\lambda+1)/D, \quad z_{ss} = -2(2\lambda+1)^3/D, \tag{6}$$

where $D = (2\lambda + 1)^2[(2\lambda + 1)^2 + 1] + 16\alpha^2$.

(e) Show that the state $(x, z) = (\sin\theta_d, \cos\theta_d)$ is a steady state of the system if the driving and feedback strengths α and λ, respectively, are set to

$$\alpha = \frac{1}{4}\sin\theta_d\cos\theta_d, \quad \lambda = -\frac{1}{2}(1 + \cos\theta_d). \tag{7}$$

(f) Show that the symmetric part of the reduced Bloch matrix $\mathbf{A}$ is negative definite if and only if $\lambda \neq -\frac{1}{2}$. Use this result to show that the distance of any state from the steady state $\|\mathbf{s}(t) - \mathbf{s}_{\rm ss}\|$ is strictly decreasing, and thus that the steady state is globally attractive, except if the target state is on the equator of the Bloch sphere.

END OF PAPER