UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2009 1:30 pm to 4:30 pm

PAPER 50

QUANTUM INFORMATION THEORY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) State the necessary and sufficient condition for a quantum error-correcting code (QECC), $\mathcal{X}$, to correct errors in a given set $\mathcal{E}$.

(b) Prove that an $[[n, k, d]]$ QECC can correct $(d-1)$ errors in known locations.

(c) Let $\mathcal{X}$ be an $[[n, k, d]]$ QECC. The quantum Singleton bound is given by

$$(n-k) \geqslant 2(d-1).$$

Prove this bound, clearly stating any inequalities that you use.

[*Hint: Let $\mathcal{X} \subset \mathcal{H}_Q$, where $\mathcal{H}_Q$ is the Hilbert space of a system Q of n qubits. If $\mathcal{X}$ is to correct arbitrary errors in a subsystem A of Q, then*

$$\rho_{RA} = \rho_R \otimes \rho_A,$$

where R is a reference system with Hilbert Space $\mathcal{H}_R \simeq \mathcal{H}_Q$.]

(d) Let $\mathcal{X}$ denote a non-degenerate $[[n, k, d]]$ QECC. Let t denote the maximum weight of a Pauli operator which can be corrected by $\mathcal{X}$. Express d in terms of t. Find the total number, $N(t)$, of possible errors of weight upto t that can occur on n qubits. The Quantum Hamming bound states that for a non-degenerate code

$$N(t) \leqslant 2^{n-k}.$$

Justify this bound.

2 (a) Prove that any completely positive trace-preserving map Φ, acting on states ρ in a Hilbert space $\mathcal{H}_A$, can be written in the Kraus form:

$$\Phi(\rho) = \sum_k A_k \rho A_k^\dagger,$$

where A_k are linear operators satisfying $\sum_k A_k^\dagger A_k = \mathbf{I}$, and $\mathbf{I}$ is the identity operator.

Hint: Consider a maximally entangled state and use the relative state method, stating clearly what is meant by a relative state and an index state.

(b) Using a maximally entangled state and the properties of the swap operator, prove that the transposition operator T is positive but not completely positive.

3 (a) Let $\mathcal{E} := \{p_i, \rho_i\}$ denote an ensemble of quantum states ρ_i, occurring with probabilities p_i. Define the Holevo χ quantity of this ensemble and prove that it can never increase under the action of a completely positive trace-preserving map on the states.

(b) Consider a bipartite quantum system AB, with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, which is in a state ρ_{AB}. Define the quantum conditional entropy $S(A|B)$. Prove that $|S(A|B)| \leqslant \log \dim \mathcal{H}_A$, carefully stating any other properties that you use.

(c) Let $\{P_i\}$ denote a complete set of orthogonal projection operators acting on the Hilbert space of a system which is in a state ρ. Prove that the von Neumann entropy, $S(\sigma)$, of the state $\sigma := \sum_i P_i \rho P_i$ of the system, is at least as large as that of the state ρ, i.e., prove that $S(\sigma) \geqslant S(\rho)$. When does the equality hold?

(d) Let $\mathcal{H}$ be a Hilbert Space of dimension d and let $\{|\phi_i\rangle\}$ be an orthonormal basis in $\mathcal{H}$. Prove that $S(\rho) \leqslant S(\rho_d)$ where $\langle\phi_i|\rho_d|\phi_j\rangle = \delta_{ij}\langle\phi_i|\rho|\phi_i\rangle$.

4 (a) Suppose Alice has two classical bits which she wants to send to Bob, who is in a different location. However, she can only use a noiseless quantum channel for this purpose. She is not allowed to communicate classically with Bob. Can she achieve her goal? What is the name of the protocol that should be used? Explain clearly what she and Bob would have to do.

(b) Can the pure state $|\psi_1\rangle := \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ be converted to the pure state $|\psi_1\rangle := \sin\theta\,|01\rangle + \cos\theta\,|10\rangle$, where $0 < \theta < \pi/4$, by local operations and classical communication only? Justify your answer.

5 Consider the decay of a two-level atom from its excited state, to its ground state. Let the probability of this decay be p. The spontaneous emission of a photon accompanies this decay.

(a) Name the quantum channel that can be used to model this process and write its Kraus operators. What process does each of these Kraus operators correspond to? Give reasons for your answer.

(b) What is a unital channel? Is the channel in **(a)** unital?

(c) Suppose the atom is originally in a state $\rho := \sum_{\alpha,\beta=0}^{1} \rho_{\alpha\beta}|\alpha\rangle\langle\beta|$, where $|\alpha\rangle$, $|\beta\rangle$, $\alpha, \beta \in \{0, 1\}$, denote orthonormal basis states of the Hilbert space of the atom. By considering the action of a unitary operator U on the atom and its environment, deduce how the state of the atom changes under the action of one use of the channel in **(a)**.

(d) Let AB denote a bipartite system which is in a state ρ_{AB}. Show that the mutual information of the system cannot increase under the action of a completely positive trace-preserving map on the subsystem B alone.

END OF PAPER