UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2009 1:30 pm to 4:30 pm

PAPER 5

PRO-P GROUPS

*Answer no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define a *topological group* and prove the following results:

(i) every open subgroup of a topological group is closed.
(ii) every open subgroup of a compact topological group is of finite index.

Define an *inverse system* and an *inverse limit* of topological spaces.

Prove that inverse limits exist and are unique.

Give two definitions of a profinite group and explain why they are equivalent. [*State results from topology as required.*]

2 Let G be a profinite group. Define the Frattini subgroup, $\Phi(G)$, of G. Let $X \subseteq G$. Prove that:

X generates G topologically if and only if $X\Phi(G)/\Phi(G)$ generates $G/\Phi(G)$ topologically.

Now, let G be a pro-p group. Stating clearly any results you use prove that $\Phi(G) = \overline{G^p[G,G]}$.

Let G be a pro-p group and K a subgroup of finite index in G. Prove that the index of K in G, $|G : K|$, is a power of p. [*Hint: Suppose* $|G : K| = m = p^r q$ *where* p *and* q *are coprime. Let* $X = \{g^m : g \in G\}$ *and* N *be an open normal subgroup of* G*. Let* $g \in G$ *and show that* $g^{p^r} \in XN$.]

Prove that if G is a finitely generated pro-p group then every subgroup of finite index in G is open. State clearly any results you use.

3 Throughout this question let p be an ODD prime.

Let G be a finite p-group and N a subgroup of G. Explain what it means to say that N is *powerfully embedded* in G and that G is *powerful.*

Prove the following:

(i) if N is powerfully embedded in G and $H = \langle x, N \rangle$ then H is powerful.

(ii) if $N = \langle X \rangle^G$ (the normal subgroup generated by X) and N is powerfully embedded in G then $N = \langle X \rangle$.

(iii) if N is powerfully embedded in G then $[N, G]$ is also powerfully embedded in G, where $[N, G]$ is the subgroup generated by commutators $[n, x]$ with $n \in N$ and $x \in G$. [*You may use the fact that it is sufficient to prove this under the assumption that* $[N, G, G, G] = 1$.]

Define the *lower p-series* $G_1 \geqslant G_2 \geqslant G_3 \geqslant \cdots$ of G. Suppose G is a powerful p-group, prove the following:

(iv) the map $x \mapsto x^p$ induces a homomorphism from G_1/G_2 onto G_2/G_3.

(v) $G^p = G^{\{p\}}$ where $G^{\{p\}} = \{g^p : g \in G\}$ and G^p is the subgroup generated by this set. [*You may use, without proof, the result that says if* N *is powerfully embedded in* G *then so is* N^p.]

Stating clearly any results you use prove the following:

(vi) a powerful 2-generator finite p-group is metacyclic (i.e. has a cyclic normal subgroup with cyclic quotient). [*Hint: show the derived group is cyclic.*]

4 Write an essay describing the meaning and proof of the following theorem:

The maps $G \mapsto L_G$ and $L \mapsto (L, *)$ are mutually inverse isomorphisms between the category of uniform pro-p groups and the category of powerful Lie algebras over $\mathbb{Z}_p$.

Make sure you define terms you use, such as, uniform pro-p groups, powerful Lie algebras, L_G and $(L, *)$.

5 Let G be a pro-p group and A a topological G-module. Explaining clearly any terms you use define the n^{th}-cohomology group $H^n(G, A)$. Prove that

$$d(G) = \dim(H^1(G, \mathbb{F}_p)),$$

where $d(G)$ denotes the minimal number of generators needed to topologically generate G. State clearly any results you use.

Define a *free pro-p group* and a *finite presentation* of a pro-p group G.

Let G be a pro-p group and K an open normal subgroup of G. Prove that if K is finitely-presented then so is G.

END OF PAPER