UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2009 9:00 am to 11:00 am

## PAPER 49

## SOLITONS AND INSTANTONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1 Solitons and Instantons**

Consider the nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - |\psi|^2\psi$$

to be solved for $\psi(x,t) \in \mathbb{C}$. Search for stationary soliton solutions $\psi(x,t) = e^{-itE}f(x)$ by obtaining an ordinary differential equation for $f(x) \in \mathbb{R}$, and solving it with the condition that $\int f^2 dx < \infty$.

Show that it is possible to boost these solutions to solitons moving at fixed velocity along a line $X = ut + X_0$, giving an explicit formula for the solution.

Consider the nonlinear Schroedinger equation in a potential $V = V(\epsilon x)$, with $\epsilon$ small,

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + V(\epsilon x)\psi - |\psi|^2\psi.$$

Explain very briefly (without doing any calculations) how you would use the method of effective Lagrangian to approximate the motion of the soliton under the influence of the potential.

For the case of the harmonic potential $V = \frac{1}{2}x^2$ show that there exist exact soliton solutions of the form $e^{i\Theta(x,t)}f(y)$ where $\Theta(x,t)$ is real, $y = x - X(t)$, with $\ddot{X} + X = 0$, and where $f(y)$ is a solution of

$$Ef = -\frac{1}{2}\frac{d^2 f}{dy^2} - f^3 + \frac{1}{2}y^2 f.$$

(This equation need not be solved explicitly. You may assume it has a solution satisfying $\int f^2 dy < \infty$.) Give a formula for the phase $\Theta(x,t)$.

**2 Solitons and Instantons**

(i) Explain the Derrick scaling argument for non-existence of finite energy static solutions for a scalar field theory in three or more space dimensions. What does the Derrick argument tell you in two space dimensions?

(ii) Consider the two dimensional static abelian Higgs model, with energy

$$V(A,\Phi) = \int_{\mathbb{R}^2} \left( B^2 + |(\nabla - iA)\Phi|^2 + \frac{1}{4}(1-|\Phi|^2)^2 \right) d^2x$$

where $\Phi(x) \in \mathbb{C}$ and $A = A_1 dx^1 + A_2 dx^2$ is the magnetic vector potential with magnetic field $B = \partial_1 A_2 - \partial_2 A_1$. Carry out the Derrick scaling argument for $V$, and show that any static soliton solution must satisfy

$$\int_{\mathbb{R}^2} B^2 d^2x = \frac{1}{4}\int_{\mathbb{R}^2} (1-|\Phi|^2)^2 d^2x.$$

(iii) Explain briefly how to construct the multi-vortex soliton solutions for the abelian Higgs model corresponding to vortices located at arbitrary locations in the plane.

(iv) Assume $(A,\Phi)$ is such that

$$\lim_{|x|\to+\infty} |\Phi(x)| = 1$$

and $B, (\nabla - iA)\Phi$ and $|1-|\Phi|^2|$ decay sufficiently rapidly as $|x| \to +\infty$. Show that

$$\int_{\mathbb{R}^2} B d^2x = \lim_{R\to+\infty} \int_{|x|=R} A_j dx^j = 2\pi N$$

for some integer $N$, and explain the significance of $N$ in terms of the field $\Phi$. What is the integer $N$ for the multi-vortex solutions you have described in part (iii)?

**3 Solitons and Instantons**

Consider the $\sigma$ - model energy

$$V(\phi) = \int_{\mathbb{R}^2} |\nabla\phi|^2 d^2x$$

for maps $\phi : \mathbb{R}^2 \to S^2 = \{\phi : \sum_{j=1}^3 \phi_j^2 = 1\}$. Use stereographic projection $S^2 \to \mathbb{C}$ to give a corresponding map $w : \mathbb{R}^2 \to \mathbb{C}$, i.e. explicitly define $w(x) = w_1(x) + iw_2(x)$ by the formula

$$w_1(x) + iw_2(x) = \frac{\phi_1(x) + i\phi_2(x)}{1 + \phi_3(x)}$$

at each $x \in \mathbb{R}^2$. Calculate the energy in terms of $w$, i.e. the energy functional $\tilde{V}(w) = V(\phi)$.

The degree of the map $\phi : \mathbb{R}^2 \to S^2$ is defined by

$$\deg\phi = \frac{1}{4\pi} \int_{\mathbb{R}^2} \phi \cdot (\partial_1\phi \times \partial_2\phi) d^2x,$$

and is an integer for smooth finite energy maps. Write the degree of the map as an integral involving $w$ and its derivatives.

Hence obtain a lower bound for $\tilde{V}(w)$ on the set of maps of fixed degree, $N \in \mathbb{Z}$. For each $N$ give an example of a map of degree $N$ for which this lower bound is achieved.

**4 Solitons and Instantons**

Consider the static Yang-Mills-Higgs functional on $\mathbb{R}^3$:

$$V_\lambda(A,\Phi) = \int_{\mathbb{R}^3} \left[ \langle F_{jk}, F_{jk} \rangle + \langle D_j\Phi, D_j\Phi \rangle + \lambda(1 - \langle \Phi, \Phi \rangle)^2 \right] d^3x$$

where $F_{jk}$ is the field associated to the gauge potential $A_j$, and $\Phi$ is the Higgs field, and all these fields take values in the Lie algebra $su(2)$, on which the inner product $\langle\, \Phi\,,\, \Psi\,\rangle = -\mathrm{trace}\,\Phi\Psi$ on $su(2)$ is used. The convariant derivative $D_j\Phi$ is defined by

$$D_j\Phi = \nabla_j\Phi + [A_j,\, \Phi].$$

Define the gauge transformations and show that $V_\lambda$ is invariant under gauge transformations.

Assuming that in some region of space $\Phi(x) = \varphi(x)e^3$ where $\varphi(x)$ is real valued, and $e^1, e^2, e^3$ form an orthonormal basis of $su(2)$, calculate the quantity

$$f_{jk} = \Big\langle F_{jk}, \frac{\Phi}{|\Phi|} \Big\rangle - \frac{1}{|\Phi|^3} \big\langle \Phi, [D_j\Phi, D_k\Phi] \big\rangle.$$

Deduce that it satisfies the equation

$$\partial_i f_{jk} + \partial_j f_{ki} + \partial_k f_{ij} = 0$$

in any region of space where $\Phi$ is non-zero.

[You may assume, without giving a proof, that for $x$ in a small ball in which $\Phi$ is never zero, it is possible to find a gauge transformation so that $\Phi(x)$ has any fixed direction in the internal space $su(2)$. The summation convention is assumed throughout the question.]

**END OF PAPER**