UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2009 9:00 am to 12:00 pm

PAPER 48

ADVANCED QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 a) Express the path integral,

$$\mathcal{I}[q^{(i)}, q^{(f)}] \quad = \quad \int_{q(0)=q^{(i)}}^{q(T)=q^{(f)}} [dq(t)] \exp\left(\frac{i}{\hbar}\int_0^T dt\ \frac{1}{2}m\dot{q}^2\right)$$

as a matrix element of quantum mechanical operators and evaluate it using standard operator methods.

Evaluate the action $S_{cl}[q^{(i)}, q^{(f)}]$ of the classical path of a free non-relativistic particle of mass m with boundary conditions $q(0) = q^{(i)}$, $q(T) = q^{(f)}$. Show that,

$$\mathcal{I}[q^{(i)}, q^{(f)}] \quad = \quad C \exp\left(\frac{i}{\hbar}S_{cl}[q^{(i)}, q^{(f)}]\right)$$

where C is a constant which does not depend on $q^{(i)}$ and $q^{(f)}$.

b) Consider the Taylor expansion of the integral,

$$I \quad = \quad \int_{-\infty}^{+\infty} dx\ \exp\left(-\frac{m^2}{2}x^2 - \frac{\lambda}{4!}x^4\right)$$

in powers of λ. Write down Feynman rules for calculating the coefficients in this expansion and draw all Feynman diagrams contributing to the term of order λ^2. Calculate the coefficient of this term by evaluating these diagrams.

2 Explain what is meant by the *Superficial Degree of Divergence*, D, of a Feynman diagram. For a diagram in scalar field theory in d spacetime dimensions with E external lines, L loops and V_n n-valent vertices, show that,

$$D \;=\; (d-4)L + \sum_n (n-4)V_n \;-\; E \;+\; 4$$

You may use Euler's formula $L = I - V + 1$, where I is the number of internal lines and $V = \sum_n V_n$ is the total number of vertices, without proof. Under what circumstances is such a theory *renormalisable*?

Consider a scalar field theory in $d = 6$ spacetime dimensions with Lagrangian density,

$$\mathcal{L} \;=\; -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \;-\; \frac{m^2}{2}\phi^2 \;-\; \frac{g}{3!}\phi^3 \qquad (*)$$

Find an expression for the superficial degree of divergence D of a Feynman diagram in this theory in terms of the number E of external lines. Is this theory renormalisable? Justify your answer.

In d spacetime dimensions, the amputated two-point function $\hat{F}_2(p_1,p_2)$is related to the full two-point Green's function $F_2(p_1,p_2)$ by,

$$F_2(p_1,p_2) \;=\; i(2\pi)^d\,\delta^{(d)}(p_1+p_2)\,\times\,\frac{-i}{p_1^2+m^2}\,\times\,\frac{-i}{p_2^2+m^2}\,\times\,\hat{F}_2(p_1,p_2)$$

Write down Euclidean momentum space Feynman rules for the six-dimensional theory with Lagrangian $(*)$ and draw the one-loop diagram contributing to the amputated two-point function $\hat{F}_2(p_1,p_2)$. Show that the divergent part of this diagram can be written in the form Ap^2+B where $p = p_1 = -p_2$ and A and B are momentum independent. Evaluate B in the presence of a momentum space cut-off Λ. You may use without proof the fact that $Vol(S^5) = \pi^3$.

3

Derive the Callan-Symanzik equation,

$$\left(\mu\frac{\partial}{\partial\mu} \;+\; \hat{\beta}_\lambda\frac{\partial}{\partial\lambda} \;+\; m^2\gamma_{m^2}\frac{\partial}{\partial m^2} \;+\; n\gamma_\phi\right)G_n \;=\; 0$$

for the n-point function $G_n(p_1,\ldots,p_n;m,\lambda,\mu)$ of ϕ^4-theory with renormalised mass and coupling m and λ respectively, defined in a renormalisation scheme which depends on an arbitrary scale μ. You should give definitions for the coefficient functions, $\hat{\beta}_\lambda$, γ_{m^2} and γ_ϕ appearing in the equation.

In dimensional regularisation with the MS renormalisation scheme, the bare and renormalised parameters of the theory in $d = 4-\varepsilon$ dimensions are related as,

$$\lambda_B \;=\; \mu^\varepsilon\left(\lambda \;+\; \sum_{n=1}^{\infty}\frac{f_n(\lambda)}{\varepsilon^n}\right)$$

$$m_B^2 \;=\; m^2\left(1 \;+\; \sum_{n=1}^{\infty}\frac{b_n(\lambda)}{\varepsilon^n}\right)$$

Show that,

$$\beta_\lambda = \hat{\beta}_\lambda + \varepsilon\lambda = \left(\lambda\frac{\partial}{\partial\lambda} - 1\right)f_1(\lambda), \qquad\qquad \gamma_{m^2} = \lambda\frac{\partial}{\partial\lambda}b_1(\lambda)$$

At one-loop the above relations take the specific form,

$$\lambda_B \;=\; \lambda\mu^\varepsilon\left(1 \;+\; \frac{3\lambda}{16\pi^2\varepsilon}\right)$$

$$m_B^2 \;=\; m^2\left(1 \;+\; \frac{\lambda}{16\pi^2\varepsilon}\right)$$

up to corrections of higher order in λ. There is no field renormalisation at this order. Calculate the corresponding values of β_λ, γ_{m^2} and γ_ϕ.

By defining an appropriate running coupling $\lambda(\mu)$, which you should determine as a function of μ, solve the Callan-Symanzik equation to one-loop accuracy in the case $m^2 = 0$.

How is your solution modified in the presence of a non-zero anomalous dimension γ_ϕ? Working with the same one-loop formula for β_λ (and $m^2 = 0$), determine the new solution explicitly in the case,

$$\gamma_\phi \;=\; c\lambda^2$$

for some constant c.

4 Write an essay on the quantization of non-abelian gauge theory. You should discuss the problems associated with quantising gauge theories and how they are resolved in the Faddeev-Popov approach. You should also outline the derivation of a gauge-fixed Lagrangian for the theory.

END OF PAPER