UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2009 1:30 pm to 4:30 pm

PAPER 47

STANDARD MODEL

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You may find the following information useful.

For Dirac matrices γ^μ:

$$Tr\,[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma] \;=\; 4\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\,,$$
$$Tr\,[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma\gamma_5] \;=\; 4\,i\,\epsilon^{\mu\nu\rho\sigma},$$

where ϵ is the Levi-Civita symbol in 4 dimensions satisfying

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\chi\eta} \;=\; -2\left(\delta^\rho_\chi\,\delta^\sigma_\eta - \delta^\rho_\eta\,\delta^\sigma_\chi\right).$$

Writing a charge conjugation operator as $\hat{C}$ and a parity reversal operator as $\hat{P}$, a Dirac spinor $\psi(x)$ transforms as

$$\hat{C}\psi(x)\hat{C}^{-1} \;=\; C\bar{\psi}(x)^t\,, \qquad\qquad \hat{P}\psi(x)\hat{P}^{-1} \;=\; \gamma^0\psi(x_P)$$

ignoring phases, where x_P is obtained from x by reversing the signs of its spatial components, and C is an anti-symmetric 4 by 4 matrix satisfying $C(\gamma^\mu)^tC^{-1} = -\gamma^\mu$.

The decay width Γ for a particle with 4-momentum p^μ decaying into n particles of 4-momenta $q^\mu_{i=1,\ldots,n}$ is

$$\Gamma \;=\; \frac{1}{2\,p^0}\int\prod_{i=1}^{n}\left(\frac{d^3q_i}{(2\pi)^3 2\,E_{q_i}}\right)|\mathcal{M}|^2(2\pi)^4\delta^4\left(p - \sum_{i=1}^{n} q_i\right),$$

where $\mathcal{M}$ is the matrix element of the decay.

The fundamental representation of the generators of $SU(N)$ may be written as traceless N by N matrices T^a, where $Tr(T^aT^b) = \frac{1}{2}\,\delta^{ab}$ and $a = 1,\ldots,N^2-1$.

1 In the muon decay process

$$\mu^-(p) \to e^-(k) + \bar{\nu}_e(q) + \nu_\mu(q'),$$

suppose the initial muon is polarized. The spin polarization may be represented by a 4-vector with components s^μ, with $s.p = 0$, where in the muon rest frame $s^\mu = (0, \mathbf{s})$, and the corresponding Dirac spinor satisfies $u(p)\overline{u}(p) = (\gamma.p + m_\mu)\frac{1}{2}(1 + \gamma_5\gamma.s)$, where m_μ is the muon mass.

Ignoring leptonic mixing, write down an expression for the three-family leptonic charged current J^μ in terms of γ matrices and leptonic Dirac spinors.

Calculate $\mathcal{M}$, the relevant matrix element for the decay obtained from the effective charged-current Lagrangian density

$$\mathcal{L}_{\text{eff}}(x) = -\frac{G_F}{\sqrt{2}} \left(J^\mu(x)^\dagger J_\mu(x) \right) .$$

Next show, neglecting neutrino masses, that

$$\sum_{\text{spins } e, \bar{\nu}_e, \nu_\mu} |\mathcal{M}|^2 = G_F^2\, q'.k\ q.(Ap + Bm_\mu s) ,$$

where A and B are integers that you should calculate.

In calculating the differential width for the decay, you may assume that

$$\int \frac{d^3q}{|\mathbf{q}|} \frac{d^3q'}{|\mathbf{q}'|} \delta^4(Q - q - q') q'^\mu q^\nu = \frac{\pi}{3} Q_\mu Q_\nu + \frac{\pi}{6} g_{\mu\nu} Q^2.$$

Show that, in the muon rest frame and neglecting the electron mass, the differential decay width for the final electron to be emitted with energy E_e into a solid angle element $d\Omega(\hat{\mathbf{k}})$ about the direction $\hat{\mathbf{k}}$ is proportional to

$$x^2 \big(3 + Cx + (Dx + E)\, \hat{\mathbf{k}}.\mathbf{s}\big)\, dx\, d\Omega(\hat{\mathbf{k}}) ,$$

where $x = 2E_e/m_\mu$, and determine the integers C, D and E. What does this result imply in terms of parity conservation in muon decay, and why?

2 The tree-level QED differential cross-section for e^+e^- scattering into $\mu^+\mu^-$ is, at centre of mass energy $\sqrt{q^2} \gg 1$ GeV^2 and in the massless muon approximation,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4q^2}\left(1+\cos^2\theta\right),$$

where θ is the angle between the electron and the muon in the centre of mass frame, α is the fine structure constant, and $d\Omega$ is the differential solid angle between the muon and the electron. Calculate the total cross-section σ in terms of α^2 and q^2.

Draw a Feynman diagram representing $e^+e^- \to$ hadrons at leading order (LO). Use the e^+e^- total cross-section derived above to find the total cross-section of $e^+e^- \to$ hadrons at leading order, including a detailed discussion of assumptions and approximations.

Draw Feynman diagrams for all the next-to-leading order (NLO) QCD corrections to $e^+e^- \to$ hadrons. Discuss the regularisation of the resulting divergences in the NLO corrections. Write an expression for $\sigma_{LO+NLO}\ /\ \sigma_{LO}$, the ratio of the cross-section calculated including the NLO corrections to the leading order cross-section σ_{LO}, in terms of the strong gauge coupling g_s and a constant A which could be derived by calculating the NLO corrections explicitly. (You need not calculate A.)

Briefly state which experimental signatures are determined by the NLO processes. Which measurement gave the first direct evidence for gluons?

3 One may write the renormalisation group equation for the gauge coupling g_i of the Standard Model group i as (no sum on i)

$$\mu \frac{dg_i}{d\mu} = \frac{\beta_i}{16\pi^2} g_i^3,$$

where μ is the renormalisation scale, and β_i is the one-loop beta-function (a real constant). Find the relationship between $\alpha_i(M_Z)$ and $\alpha_i(\mu)$, where $\alpha_i = g_i^2/(4\pi)$.

Grand Unified Theories predict that at some scale $\mu = M_{GUT}$,

$$\frac{5}{3}\alpha_1(M_{GUT}) = \alpha_2(M_{GUT}) = \alpha_3(M_{GUT}).$$

Assuming this, derive the relationship

$$\alpha_3^{-1}(M_Z) = \alpha_2^{-1}(M_Z) + \frac{\beta_3 - \beta_2}{3\beta_1/5 - \beta_2}\left[\frac{3}{5}\alpha_1^{-1}(M_Z) - \alpha_2^{-1}(M_Z)\right].$$

Write down a table giving the field content of one family of the Standard Model and a Higgs, detailing for each field the irreducible representation under the complete gauge group and the spin, including the chirality.

For gauge group $SU(N)$,

$$\beta_N = -\frac{11N}{3} + \frac{2}{3}\sum_f T(f) + \frac{1}{3}\sum_s T(s),$$

where the sums are over the two-component Weyl fermions f and the complex scalars s coupled to the $SU(N)$ gauge field, and T is the appropriate Dynkin index, which is 1/2 for the fundamental representation. For gauge group $U(1)_Y$,

$$\beta_1 = \frac{2}{3}\sum_f Y_f^2 + \frac{1}{3}\sum_s Y_s^2$$

where $Y_{f,s}$ is the hypercharge of a Weyl fermion or complex scalar, respectively. Hence calculate $\beta_1, \beta_2, \beta_3$ of the Standard Model, indicating where each contribution comes from.

4 Charge conjugation C, parity reversal P, and their combination CP, play an important rôle in the Standard Model. State an example of a physical process where CP violation has been experimentally observed.

How do the following Standard Model fields transform under CP:
(a) a W-boson field $W^\mu(x)$,
(b) a conjugated up-quark field $\bar{u}(x)$,
(c) the Higgs doublet $\phi(x)$?

The quark–W couplings are, for primed quarks in the weak eigenbasis:

$$\mathcal{L}_{q'W}(x) = -\sum_i \frac{g}{2\sqrt{2}} \bar{u}'_i(x)\gamma^\mu(1-\gamma_5)d'_i(x)W_\mu(x) + h.c.$$

Describe how the weak eigenbasis quarks are related to the (unprimed) mass eigenstates. Re-write $\mathcal{L}_{q'W}(x)$ in terms of mass eigenstate up- and down-quark fields u_i, d_i respectively, thus explaining how the CKM matrix V_{ij} arises. Perform a CP transformation on $\mathcal{L}_{q'W}$ and thus show that CP conservation in charged current interactions would require that the matrix V_{ij} is real.

In a Standard Model with N families of quarks, calculate the number of *physical* CP-violating phases required to parametrise V_{ij}.

Denoting a gluon field as $A^a_\mu(x)$ and the corresponding $SU(3)$ generators in the fundamental representation as T^a, write down the covariant derivative D_μ acting on an up-quark u. Demanding that the gluon–up-quark interaction be invariant under CP, derive an expression for how $A^a_\mu(x)T^a$ transforms under CP. The gluon field strength may be written as

$$F^a_{\mu\nu}(x)T^a = -\frac{i}{g}[D_\mu, D_\nu],$$

Calculate the CP transformation of $F^a_{\mu\nu}(x)T^a$ and hence determine whether a Lagrangian term

$$\mathcal{L}_\theta(x) = \theta\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x),$$

where θ is a real constant, conserves CP.

END OF PAPER