UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2009 1:30 pm to 4:30 pm

PAPER 46

STRING THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Polyakov action for a string in flat space is given by

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, \eta_{\mu\nu}$$

Describe the symmetries of this action and give the transformation for the fields. Explain how these symmetries can be used to reduce the theory to that of free scalar fields,

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma\; \partial_\alpha X^\mu \partial^\alpha X^\nu\, \eta_{\mu\nu}$$

together with some constraints. What are these constraints?

Show that the Noether charge associated to spacetime translations is

$$P^\mu = \frac{1}{2\pi\alpha'} \int d\sigma\; \frac{dX^\mu}{d\tau}$$

Show that the Noether charge associated to spacetime Lorentz transformations is

$$J^{\mu\nu} = \frac{1}{2\pi\alpha'} \int d\sigma\; \left(X^\mu \frac{dX^\nu}{d\tau} - X^\nu \frac{dX^\mu}{d\tau} \right)$$

A closed string is parameterized by $\sigma \in [0, 2\pi]$ and $\tau \in \mathbb{R}$. Show that the configuration

$$X^0 = R\tau \quad , \quad X^1 = R\cos\tau\cos\sigma \quad , \quad X^2 = R\cos\tau\sin\sigma$$

solves the equations of motion and the constraints. What does this solution look like in spacetime?

Compute the spacetime momentum P^μ and angular momentum $J^{\mu\nu}$ for this solution. Use this to show that the mass per unit length of the string is $1/2\pi\alpha'$.

2 What is meant by a quasi-primary operator of weight $(h, \tilde{h})$ in a conformal field theory? What is a primary operator? What is a primary state?

Describe the state-operator map. How does this relate primary states to primary operators. [You will need to use the fact that the Virasoro generators are given by a contour integral of the stress-tensor $T(z)$,

$$L_n = \frac{1}{2\pi i} \oint dz\ z^{n+1}\, T(z)$$

with a similar relationship between $\tilde{L}_n$ and $\bar{T}(\bar{z})$.]

Explain why the spectrum of string theory is related to primary operators of weight $(1, 1)$. How does this requirement determine the mass of string states?

Two commuting complex fields β and γ obey the equation of motion $\bar{\partial}\beta = \bar{\partial}\gamma = 0$. Their operator product expansion is given by

$$\beta(z)\gamma(w) = \gamma(w)\beta(z) = -\frac{1}{z-w} + \ldots$$

with no singular terms in the $\beta(z)\beta(w)$ and $\gamma(z)\gamma(w)$ expansions. Consider the one-parameter family of stress-energy tensors

$$T =: (\partial\beta)\gamma : -\lambda\, \partial : \beta\gamma : \quad , \quad \bar{T} = 0$$

with $\lambda \in \mathbb{R}$. Show that β is primary with weight $(\lambda, 0)$ and γ is primary with weight $(1 - \lambda, 0)$. Show that the central charge of this system is equal to

$$c = 12\lambda^2 - 12\lambda + 2$$

3 Explain how the dependence of scattering amplitudes on the string coupling g_s arises in the path integral formulation of the closed string.

The scattering amplitude for four tachyonic ground states of the closed bosonic string is given by the Euclidean path integral

$$\mathcal{A}^{(4)} = \frac{g_s^2}{\mathrm{Vol}(SL(2;\mathbb{C}))} \int \prod_{i=1}^{4} d^2 z_i \int \mathcal{D}X \; e^{-S[X]} \, V(z_1, p_1) \ldots V(z_4, p_4)$$

Here $V(z,p) = e^{ip\cdot X}$ and the action is that of D free scalar fields X^μ,

$$S = \frac{1}{2\pi\alpha'} \int d^2 z \; \partial X^\mu \, \bar{\partial} X^\nu \, \delta_{\mu\nu}$$

What is the propagator for the scalar fields X^μ? Show that the functional integral can be reduced to the form

$$\mathcal{A}^{(4)} \sim \frac{g_s^2}{\mathrm{Vol}(SL(2;\mathbb{C}))} \int \prod_{i=1}^{4} d^2 z_i \, \prod_{j<l} |z_j - z_l|^{\alpha' p_j \cdot p_l}$$

When the momenta are on-shell, so that $p_i^2 = 4/\alpha'$, show that the integrand is invariant under the $SL(2;\mathbb{C})$ transformation

$$z_i \to \frac{a z_i + b}{c z_i + d}$$

where a, b, c and $d \in \mathbb{C}$ and $ad - bc = 1$.

The four-point tree-level amplitude for the scattering of massless particles in the Type II superstring includes the factor

$$\mathcal{A}^{(4)} \sim g_s^2 \; \frac{\Gamma(-\alpha' s/4)\Gamma(-\alpha' t/4)\Gamma(-\alpha' u/4)}{\Gamma(1+\alpha' s/4)\Gamma(1+\alpha' t/4)\Gamma(1+\alpha' u/4)}$$

where s, t and u are the usual Mandelstam variables. The gamma function, $\Gamma(x)$, has simple poles at $x = 0, -1, -2, \ldots$ and no other singularities in the complex plane. By analyzing the poles in the s-channel, what can you say about the mass spectrum of the Type II superstring?

4 Describe *in outline* how the quantization of the two-dimensional worldsheet theory describing a closed bosonic string moving in background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ leads to an action in $D = 26$ dimensions. [Concentrate on conceptual ideas rather than mathematical details.]

A Dp-brane of tension T_p moving in these background fields is described by the action

$$S = -T_p \int d^{p+1}\xi \; e^{-\tilde{\Phi}} \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab} + B_{ab})}$$

where a, $b = 0, \ldots, p$. Explain how γ_{ab}, B_{ab} and $\tilde{\Phi}$ are related to $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. Provide heuristic arguments for this form of the action.

Explain why two coincident D-branes give rise to a $U(2)$ non-Abelian gauge symmetry. The massless excitations of the branes are a Yang-Mills gauge potential with field strength F_{ab} and adjoint scalar fields ϕ^I. In what limit is the dynamics of these fields governed by the action

$$S = -(2\pi\alpha')^2 T_p \int d^{p+1}\xi \; \mathrm{Tr}\left(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\mathcal{D}_a\phi^I\mathcal{D}^a\phi^I + \frac{1}{4}\sum_{I\neq J}[\phi^I, \phi^J]^2\right)$$

where $\mathcal{D}_a\phi^I = \partial\phi^I/\partial\xi^a + i[A_a, \phi^I]$ and $I = 1, \ldots, D-p-1$.

How are the positions of the D-branes described in terms of these fields? Why does separating the D-branes break the gauge symmetry to $U(1) \times U(1)$? Compute the mass of the W-boson. By setting this equal to the classical mass of a string stretched between the branes, determine the exact relation between the expectation values of ϕ^I and the transverse positions X^I of the D-branes.

END OF PAPER