UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2009 1:30 pm to 4:30 pm

PAPER 45

SYMMETRY AND PARTICLES

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe the relation between the groups $SU(2)$ and $SO(3)$. For $SU(2)$ define a spinor η^α, $\alpha = 1, 2$, such that it transforms as

$$\eta^\alpha \underset{A}{\longrightarrow} \eta'^\alpha = A^\alpha{}_\beta \eta^\beta\,, \qquad A \in SU(2)\,.$$

Show how to define the conjugate spinor $\bar\eta_\alpha$ and describe how it transforms. What are the invariant tensors for $SU(2)$?

Describe how irreducible representations of $SU(2)$ may be obtained by considering a representation space $\mathcal{V}_n$ formed by totally symmetric tensors $S^{\alpha_1\ldots\alpha_n}$. What is the dimension of $\mathcal{V}_n$? Do these representations give representations for $SO(3)$?

For two such spaces $\mathcal{V}_m$ and $\mathcal{V}_n$, with tensors $S^{\alpha_1\ldots\alpha_m}$ and $T^{\alpha_1\ldots\alpha_n}$, show how their product can be decomposed in terms of irreducible tensors. Demonstrate also how the product of three spinors ${\eta_1}^\alpha, {\eta_2}^\beta, {\eta_3}^\gamma$ can be decomposed in terms of irreducible tensors.

Briefly describe how this is related to the possible spin states of baryons formed from three quarks.

2 For a Lie algebra with a basis $\{T_a\}$ and commutators $[T_a, T_b] = f^c{}_{ab} T_c$ define the adjoint representation. Define the Killing form κ_{ab} and show that it satisfies

$$\kappa_{dc} f^d{}_{ab} + \kappa_{bd} f^d{}_{ac} = 0\,.$$

For a field ϕ belonging to a representation space for the Lie algebra, so that there are associated matrices $\{t_a\}$ satisfying $[t_a, t_b] = f^c{}_{ab} t_c$, explain how the covariant derivative

$$D_\mu \phi = \partial_\mu \phi + A^a{}_\mu t_a \phi\,,$$

transforms under local infinitesimal gauge transformations. Define the corresponding field strength $F^a{}_{\mu\nu}$. How does this transform under infinitesimal gauge transformations? If

$$\mathcal{L} = -\tfrac{1}{4} g_{ab}\, F^{a\mu\nu} F^b{}_{\mu\nu}\,,$$

is a gauge invariant Lagrangian what properties must g_{ab} satisfy?

If the Lie algebra is simple explain why the Killing form provides an essentially unique solution for g_{ab}.

3 The Poincaré group is defined by transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \,,$$

such that $g_{\mu\nu}(x_1 - x_2)^\mu (x_1 - x_2)^\nu$, with $g_{\mu\nu}$ the Minkowski metric, is invariant. Denoting the group elements by (Λ, a) determine the group multiplication rule. Show how rotations form a subgroup. Define a Lorentz boost. Why do boosts not form a subgroup in general?

Suppose $U[\Lambda, a]$ are unitary operators which give a representation of the Poincaré group when $\Lambda^0{}_0 \geqslant 1$, $\det \Lambda = 1$. Show how irreducible representations are obtained in terms of states

$$|p, s\, s_3\rangle \,, \quad g^{\mu\nu} p_\mu p_\nu = m^2 \,, \quad p_0 > 0 \,, \quad s_3 = -s, -s+1, \ldots, s \,,$$

for any $m^2 > 0$ and $s \in \{0, \frac{1}{2}, 1, \ldots\}$.

Suppose the symmetry is extended to include invariance under spatial reflections, with an associated unitary operator $\mathcal{P}$. How does $\mathcal{P}$ act on the states forming the representation?

4 For the Lie algebra $\{E_\pm, H\}$ with commutators

$$[E_+, E_-] = H \,, \qquad [H, E_\pm] = \pm 2 E_\pm \,, \qquad (*)$$

show how a basis for a finite dimensional representation space $\mathcal{V}_n$ may be obtained starting from a state $|n\rangle$ which satisfies $E_+|n\rangle = 0$, $H|n\rangle = n|n\rangle$ where $n = 0, 1, 2. \ldots$. What is the dimension of $\mathcal{V}_n$?

A basis for the Lie algebra of $SU(3)$ is given by $\{T^i{}_j : i, j = 1, 2, 3; T^i{}_i = 0\}$ with commutation relations

$$[T^i{}_j, T^k{}_l] = \delta^k{}_j T^i{}_l - \delta^i{}_l T^k{}_j \,.$$

Show that $E_{1+} = T^1{}_2$, $E_{1-} = T^2{}_1$, $H_1 = T^1{}_1 - T^2{}_2$ and also $E_{2+} = T^2{}_3$, $E_{2-} = T^3{}_2$, $H_2 = T^2{}_2 - T^3{}_3$ both satisfy the commutation relations in $(*)$.

Starting from a state $|n_1, n_2\rangle$ which satisfies $E_{i+}|n_1, n_2\rangle = 0$, $H_i|n_1, n_2\rangle = n_i|n_1, n_2\rangle$, for $i = 1, 2$, describe how a basis for a $SU(3)$ representation space may be obtained for the cases $(n_1, n_2) = (1, 0), (1, 1), (3, 0)$. Plot the allowed states on a diagram with x coordinate given by the eigenvalues of H_1, the y coordinate by the eigenvalue of $\frac{1}{\sqrt{3}}(H_1 + 2H_2)$. What are the dimensions of the spaces in each of these examples?

Describe briefly how these representation spaces are relevant in particle physics.

END OF PAPER